De Broglie-Bohm Prediction of Quantum Violations for Cosmological Super-Hubble Modes

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The hypothesis of quantum nonequilibrium at the big bang is shown to have observable consequences. For a scalar field on expanding space, we show that relaxation to quantum equilibrium (in de Broglie-Bohm theory) is suppressed for field modes whose quantum time evolution satisfies a certain inequality, resulting in a 'freezing' of early quantum nonequilibrium for these particular modes. For an early radiation-dominated expansion, the inequality implies a corresponding physical wavelength that is larger than the (instantaneous) Hubble radius. These results make it possible, for the first time, to make quantitative predictions for nonequilibrium deviations from quantum theory, in the context of specific cosmological models. We discuss some possible consequences: corrections to inflationary predictions for the cosmic microwave background, non-inflationary super-Hubble field correlations, and relic nonequilibrium particles.

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1 Introduction

Hidden-variables theories, such as the pilot-wave theory of de Broglie [1, 2] and Bohm [3], reproduce quantum theory for a particular 'equilibrium' distribution of hidden parameters. But allowing arbitrary distributions (analogous to non-thermal distributions in classical physics) opens up the possibility of new, 'nonequilibrium' physics that lies outside the domain of quantum physics [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Such new physics may have existed in the very early universe, with relaxation to quantum equilibrium having taken place during the violence of the big bang [4, 5, 6, 7, 8, 15]. In this paper, the hypothesis of early quantum nonequilibrium is shown to have observable consequences today.

The concept of quantum nonequilibrium has been discussed for general (deterministic) hidden-variables theories [9, 10, 12, 14]. For the specific case of de Broglie-Bohm theory, it amounts to having configurations with a distribution P that differs from the usual Born-rule distribution $|\Psi|^2$ (for a pure subensemble with wave function Ψ) [4, 5, 6, 14]. There were several motivations for proposing that the early universe began in a state of quantum nonequilibrium. Let us briefly summarise them.

There seems to be a peculiar 'conspiracy' at the heart of modern physics, whereby quantum nonlocality cannot be used to send practical instantaneous signals. In hidden-variables theories, this conspiracy is explained as a contingency of the quantum equilibrium state. Nonlocal signalling is generally possible out of equilibrium (suggesting the existence of an underlying preferred foliation of spacetime [16]); whereas in equilibrium, nonlocal effects cancel out at the statistical level [5, 6, 9, 10]. Our inability to convert entanglement into practical nonlocal signals is then not a law of physics, but a contingency of the equilibrium state. Similarly, standard uncertainty-principle limitations on measurements are also contingencies of equilibrium [5, 6, 11]. There is a parallel here with the classical thermodynamic heat death: in the complete absence of temperature differences, it would be impossible to convert heat into work, and yet such a limitation would be a mere contingency of the state, and not a law of physics.

Furthermore, it has been shown that relaxation towards quantum equilibrium occurs, in pilot-wave dynamics, in similar fashion to thermal relaxation in classical dynamics (under analogous conditions and with similar caveats) [4, 6, 8, 17, 18]. Given that all physical systems to which we have access have undergone a long and violent astrophysical history, it is then possible to understand the ubiquitous quantum noise we see around us as, in effect, a remnant of the big bang.

On this view, the effectively local and indeterministic quantum physics we experience today emerged via relaxation processes (presumably occurring close to the big bang) out of a fundamentally nonlocal and deterministic physics — a physics whose details are currently screened off from view, by the all-pervading statistical noise. For as equilibrium is approached, the possibility of instantaneous signalling disappears, and statistical uncertainty emerges. In effect, a

hidden-variables analogue of the classical heat death has actually occurred in our universe, explaining the above 'conspiracy'.

The assumption of early quantum nonequilibrium was also proposed as a possible alternative resolution of the cosmological horizon problem (which persists even in some inflationary models [19]): the resulting early nonlocality might explain the otherwise puzzling homogeneity of the universe at early times [5, 6, 7, 10].

The search for early quantum nonequilibrium may also be motivated simply on the grounds that de Broglie-Bohm theory (and indeed any deterministic hidden-variables theory) certainly *allows* nonequilibrium to occur. We have an alternative formulation of quantum physics, which yields standard quantum theory in the equilibrium limit, and which yields departures from standard quantum theory outside that limit. It seems natural to explore this possible new physics, and in particular to test for it experimentally, as far as one can. If nothing else, setting experimental bounds on the existence of quantum nonequilibrium can provide new bounds on possible deviations from quantum theory [15].

Finally, if hidden-variables theories are taken seriously, one is obliged to take the possibility of nonequilibrium seriously as well: for it is only in nonequilibrium that the underlying details become visible. If the world were always and everywhere in quantum equilibrium, the details of de Broglie-Bohm trajectories (for example) would be forever shielded from experimental test. De Broglie-Bohm theory as a whole would then be unacceptable as a scientific theory. And much the same could be said for hidden-variables theories in general.

Given the above motivations, the idea that the universe relaxed to quantum equilibrium from an earlier nonequilibrium state is plausible enough. However, to be a scientific theory it is essential to make new, quantitative predictions. The new physics of systems in quantum nonequilibrium has been explored in some detail [5, 6, 8, 9, 11, 13, 14, 15], and a specific signature of nonequilibrium has been developed [12, 14]. It has also been shown that if nonequilibrium were present at the beginning of an inflationary phase, then there would be observable consequences for the statistics of the temperature anisotropies imprinted on the cosmic microwave background (CMB) [15, 20]. Further, heuristic arguments have been given, suggesting that relaxation might be suppressed for long-wavelength field modes on expanding space [15] (a suggestion that forms the starting point for the present work); and that, if relic cosmological particles decoupled sufficiently early, they might still be in nonequilibrium today [8, 15]. However, so far, no definite quantitative predictions have been made. The aim of this paper is to fill this gap.

For the first time, given a specific cosmological model, we are able to point to precisely where quantum nonequilibrium could be found. We accomplish this by studying the evolution of nonequilibrium distributions for a scalar field on expanding space. We show that relaxation is suppressed for field modes whose quantum time evolution satisfies a certain inequality. For these particular modes, early quantum nonequilibrium is 'frozen'. For a radiation-dominated expansion, the inequality implies a physical wavelength larger than the (instantaneous) Hubble radius. On the basis of these results, it is possible to make

quantitative predictions for nonequilibrium deviations from quantum theory, in the context of a given cosmological model. As we shall see, there are a number of possible consequences: in particular, infra-red corrections to inflationary predictions for the CMB, and relic nonequilibrium particles at low energies.

2 De Broglie-Bohm Scalar Field on Expanding Space

For simplicity we consider a flat metric,

$$d\tau^2 = dt^2 - a^2 d\mathbf{x}^2 \ , \tag{1}$$

where $a(t)$ is the scale factor, $H \equiv \dot{a}/a$ is the Hubble parameter, and H^{-1} is the Hubble radius. As is customary, we take $a_0 = 1$ today (at time t_0), so that $|d\mathbf{x}|$ is a comoving distance (or proper distance today). At time t, field modes have physical wavelengths $\lambda_{\mathrm{phys}} = a(t)\lambda$, where $\lambda = 2\pi/k$ is a comoving wavelength (or proper wavelength today) and $k = |\mathbf{k}|$ is the comoving wave number.

We consider a free (minimally-coupled) massless scalar field ϕ with a Lagrangian density $\mathcal{L} = \frac{1}{2} g^{1/2} \partial_\alpha \phi \partial^\alpha \phi$ or

$$\mathcal{L} = \tfrac{1}{2} a^3 \dot{\phi}^2 - \tfrac{1}{2} a (\nabla\phi)^2 \ . \tag{2}$$

The action is $\int dt \int d^3\mathbf{x}\ \mathcal{L}$. We then have a canonical momentum density $\pi = \partial\mathcal{L}/\partial\dot{\phi} = a^3\dot{\phi}$ and a Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2} \frac{\pi^2}{a^3} + \tfrac{1}{2} a (\nabla\phi)^2 \ . \tag{3}$$

Here, it is convenient to write the dynamics in Fourier space. Expressing $\phi(\mathbf{x})$ in terms of its Fourier components

$$\phi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{x}\ \phi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \ ,$$

and writing

$$\phi_{\mathbf{k}} = \frac{\sqrt{V}}{(2\pi)^{3/2}} \left(q_{\mathbf{k}1} + i q_{\mathbf{k}2} \right)$$

for real $q_{\mathbf{k}r}$ ($r = 1$, 2), where V is a box normalisation volume, the Lagrangian $L = \int d^3\mathbf{x}\ \mathcal{L}$ becomes

$$L = \sum_{\mathbf{k}r} \frac{1}{2} \left(a^3 \dot{q}_{\mathbf{k}r}^2 - a k^2 q_{\mathbf{k}r}^2 \right) \ .$$

(For $V \to \infty$, $\frac{1}{V}\sum_{\mathbf{k}} \to \frac{1}{(2\pi)^3}\int d^3\mathbf{k}$ and $V\delta_{\mathbf{k}\mathbf{k}'} \to (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$. Since ϕ is real, we have $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$ or $q_{\mathbf{k}1} = q_{-\mathbf{k}1}$, $q_{\mathbf{k}2} = -q_{-\mathbf{k}2}$. A sum over physical degrees

of freedom should be restricted to half the values of $\mathbf{k}$, for example $k_z > 0$.) Introducing the canonical momenta

$$\pi_{\mathbf{k}r} \equiv \frac{\partial L}{\partial \dot{q}_{\mathbf{k}r}} = a^3 \dot{q}_{\mathbf{k}r} \ ,$$

the Hamiltonian $H = \int d^3\mathbf{x}\ \mathcal{H}$ becomes

$$H = \sum_{\mathbf{k}r} H_{\mathbf{k}r} \ ,$$

with

$$H_{\mathbf{k}r} = \frac{1}{2a^3}\pi_{\mathbf{k}r}^2 + \frac{1}{2}ak^2 q_{\mathbf{k}r}^2 \ .$$

Pilot-wave field theory is defined in terms of the functional Schrödinger picture, with a preferred foliation of spacetime [3, 6, 7, 21, 22, 23, 24, 25, 26]. Here, the Schrödinger equation for $\Psi = \Psi[q_{\mathbf{k}r}, t]$ is

$$i\frac{\partial \Psi}{\partial t} = \sum_{\mathbf{k}r}\left(-\frac{1}{2a^3}\frac{\partial^2}{\partial q_{\mathbf{k}r}^2} + \frac{1}{2}ak^2 q_{\mathbf{k}r}^2\right)\Psi \ , \tag{4}$$

which implies the continuity equation

$$\frac{\partial \left|\Psi\right|^2}{\partial t} + \sum_{\mathbf{k}r}\frac{\partial}{\partial q_{\mathbf{k}r}}\left(\left|\Psi\right|^2 \frac{1}{a^3}\frac{\partial S}{\partial q_{\mathbf{k}r}}\right) = 0 \tag{5}$$

and the de Broglie velocities

$$\frac{dq_{\mathbf{k}r}}{dt} = \frac{1}{a^3}\frac{\partial S}{\partial q_{\mathbf{k}r}} \tag{6}$$

(where $\Psi = \left|\Psi\right| e^{iS}$). The 'pilot wave' Ψ is interpreted as a physical field in configuration space, guiding the time evolution of an individual field $\phi(\mathbf{x}, t)$ in 3-space. (Note that a similar construction may be given in any globally-hyperbolic spacetime, by choosing a preferred foliation [13], so there is no need for spatial homogeneity.)

Over an ensemble of field configurations guided by the same pilot wave Ψ, there will be some (in principle arbitrary) initial distribution $P[q_{\mathbf{k}r}, t_i]$, whose time evolution $P[q_{\mathbf{k}r}, t]$ will be determined by

$$\frac{\partial P}{\partial t} + \sum_{\mathbf{k}r}\frac{\partial}{\partial q_{\mathbf{k}r}}\left(P\frac{1}{a^3}\frac{\partial S}{\partial q_{\mathbf{k}r}}\right) = 0 \ . \tag{7}$$

If $P[q_{\mathbf{k}r}, t_i] = \left|\Psi[q_{\mathbf{k}r}, t_i]\right|^2$, then $P[q_{\mathbf{k}r}, t] = \left|\Psi[q_{\mathbf{k}r}, t]\right|^2$ for all t, and one obtains empirical agreement with standard quantum field theory [3, 22, 23, 24, 26, 25]. On the other hand, for an initial nonequilibrium distribution $P[q_{\mathbf{k}r}, t_i] \neq \left|\Psi[q_{\mathbf{k}r}, t_i]\right|^2$, for as long as P remains in nonequilibrium the predicted statistics will generally differ from those of quantum field theory. In any case, whatever the distribution P may be (equilibrium or nonequilibrium), its time evolution will be given by (7).

3 Preliminary Discussion for a Decoupled Mode

A proper treatment of nonequilibrium freezing is given in sections 4 and 5. As we shall see, our treatment is applicable to arbitrary (entangled, mixed, and interacting) quantum states. As a preliminary exercise, in this section we shall discuss some elementary features for the simple case of a single decoupled mode $\mathbf{k}$ of a free field in a pure quantum state.

From equations (4), (6), and writing $\Psi = \psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)\varkappa$ where $\varkappa$ depends only on degrees of freedom for modes $\mathbf{k}' \neq \mathbf{k}$, we find that the wave function $\psi_{\mathbf{k}}$ of a decoupled mode $\mathbf{k}$ satisfies

$$i\frac{\partial \psi_{\mathbf{k}}}{\partial t} = -\frac{1}{2a^3}\left(\frac{\partial^2}{\partial q_{\mathbf{k}1}^2} + \frac{\partial^2}{\partial q_{\mathbf{k}2}^2}\right)\psi_{\mathbf{k}} + \frac{1}{2}ak^2\left(q_{\mathbf{k}1}^2 + q_{\mathbf{k}2}^2\right)\psi_{\mathbf{k}} \ , \tag{8}$$

while the de Broglie velocities for the mode amplitudes $(q_{\mathbf{k}1}, q_{\mathbf{k}2})$ are

$$\dot{q}_{\mathbf{k}1} = \frac{1}{a^3}\frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}1}}, \qquad \dot{q}_{\mathbf{k}2} = \frac{1}{a^3}\frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}2}} \tag{9}$$

(with $\psi_{\mathbf{k}} = |\psi_{\mathbf{k}}|\, e^{is_{\mathbf{k}}}$). The time evolution of the marginal distribution $\rho_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ will then be given by

$$\frac{\partial \rho_{\mathbf{k}}}{\partial t} + \sum_{r=1,\ 2}\frac{\partial}{\partial q_{\mathbf{k}r}}\left(\rho_{\mathbf{k}}\frac{1}{a^3}\frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}r}}\right) = 0 \ . \tag{10}$$

Equations (8)–(10) are identical to those of pilot-wave dynamics for an ensemble of nonrelativistic particles of time-dependent 'mass' $m = a^3$ moving in the $q_{\mathbf{k}1} - q_{\mathbf{k}2}$ plane in a harmonic oscillator potential with time-dependent angular frequency $\omega = k/a$. We may then discuss relaxation (and relaxation suppression) for a decoupled field mode in terms of relaxation (and relaxation suppression) for a nonrelativistic two-dimensional harmonic oscillator.

Before doing so, let us recall what is already known about relaxation in pilot-wave dynamics.

For a system with configuration q and wave function ψ, the H-function

$$H = \int dq\ \rho \ln(\rho/\left|\psi\right|^2) \tag{11}$$

(the relative negentropy of an arbitrary distribution ρ with respect to $\left|\psi\right|^2$) obeys a coarse-graining H-theorem similar to the classical one [4, 6, 8]. Introducing a coarse-graining in configuration space, and assuming appropriate initial conditions for ρ and ψ, the coarse-grained function $\bar{H}(t)$ will begin to decrease with time, corresponding to an evolution of the coarse-grained density $\bar{\rho}$ towards $\overline{\left|\psi\right|^2}$. This 'subquantum H-theorem' formalises a simple intuitive idea: because ρ and $\left|\psi\right|^2$ obey the same continuity equation, they behave like two classical fluids that are 'stirred' by the same velocity field, thereby tending to become indistinguishable on a coarse-grained level.

Such relaxation has been studied numerically, on a static spacetime background, for simple one- and two-dimensional systems [6, 8, 17, 18]. For an ensemble of nonrelativistic particles in a two-dimensional box, with a wave function consisting of a superposition of the first 16 modes, it was found that relaxation occurs very efficiently, with an approximately exponential decay $\bar{H}(t) \approx \bar{H}_0 e^{-t/t_c}$ of the coarse-grained H-function (over a timescale t_c) [17]. Similar results have been obtained for an ensemble of nonrelativistic particles in a two-dimensional harmonic oscillator potential [18]. As discussed in ref. [17], the numerical timescale t_c was found to be in approximate agreement with a theoretical relaxation timescale τ defined by $1/\tau^2 \equiv -(1/\bar{H})d^2\bar{H}/dt^2$ [6]. For a particle of mass m, and using a sufficiently small coarse-graining length ε, a rough order-of-magnitude estimate yields $\tau \sim 1/\left(\varepsilon m^{1/2}(\Delta E)^{3/2}\right)$, where ΔE is the quantum energy spread associated with ψ [8, 17]. (The quantity τ is analogous to the scattering time of classical kinetic theory: one expects a significant approach to equilibrium over timescales of order τ.) If we choose a 'natural' value $\varepsilon \sim 1/\Delta p$, where Δp is the quantum momentum spread, then taking $\Delta E \sim (\Delta p)^2/2m$ one has the simple (and rough) result

$$\tau \sim \Delta t \equiv 1/\Delta E \ , \tag{12}$$

where Δt is the quantum timescale over which the wave function ψ evolves.

3.1 Relaxation for Sub-Hubble Modes in the Minkowski Limit

One expects that in the short-wavelength limit, $\lambda_{\mathrm{phys}} << H^{-1}$, the above equations (8)–(10) will reduce to those for a decoupled mode $\mathbf{k}$ on Minkowski spacetime, because (roughly speaking) the timescale $\Delta t \propto \lambda_{\mathrm{phys}}$ over which $\psi_{\mathbf{k}} = \psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ evolves will be much smaller than the expansion timescale $H^{-1} \equiv a/\dot{a}$ [15].

To obtain a more precise and rigorous statement, note first that at any time t the Hamiltonian $\hat{H}(t)$ appearing in the Schrödinger equation (8) has the same eigenfunctions and eigenvalues as are usually obtained for a two-dimensional harmonic oscillator of (instantaneous) mass $m = a^3$ and angular frequency $\omega = k/a$. Thus, for quantum numbers $n_1,\ n_2 = 0, 1, 2,\ ...$, we have energy eigenfunctions $\phi_{n_1}(q_{\mathbf{k}1}, t)\phi_{n_2}(q_{\mathbf{k}2}, t)$ and eigenvalues $E_{\mathbf{k}}(t) = (1 + n_1 + n_2)\omega(t)$. (The time dependence in $\phi_{n_1}(q_{\mathbf{k}1}, t)$ and $\phi_{n_2}(q_{\mathbf{k}2}, t)$ comes, of course, from the time dependence of $m = a^3$ and $\omega = k/a$.) The wave function at any time t may then be expanded in terms of these energy eigenstates,

$$\psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t) = \sum_{n_1,\ n_2} c_{n_1,n_2}(t)\phi_{n_1}(q_{\mathbf{k}1}, t)\phi_{n_2}(q_{\mathbf{k}2}, t) \ ,$$

and the quantum energy spread $\Delta E_{\mathbf{k}} \equiv \sqrt{\langle E_{\mathbf{k}}^2\rangle - \langle E_{\mathbf{k}}\rangle^2}$ will be

$$\Delta E_{\mathbf{k}} = \Delta n_{\mathbf{k}} \cdot \omega \ ,$$

where $n_{\mathbf{k}} \equiv n_1 + n_2$. If we consider a subsequent evolution over a time $\delta t << H^{-1}$, where H^{-1} is the timescale over which the Hamiltonian $\hat{H}(t)$ changes, then the Hamiltonian (together with its eigenfunctions and eigenvalues) will be almost constant during $(t, t+\delta t)$, and in this interval the wave function $\psi_{\mathbf{k}}$ will evolve like that of a conventional two-dimensional oscillator, with an evolution timescale

$$\Delta t \equiv \frac{1}{\Delta E_{\mathbf{k}}} = \frac{1}{\Delta n_{\mathbf{k}}} \frac{1}{\omega} = \frac{1}{\Delta n_{\mathbf{k}}} \frac{\lambda_{\mathrm{phys}}}{2\pi}$$

(where we have $\hbar = 1$). Significant evolution of $\psi_{\mathbf{k}}$ over the interval $(t, t+\delta t)$ can occur only if $\Delta t << H^{-1}$ or

$$\lambda_{\mathrm{phys}} << \Delta n_{\mathbf{k}} \cdot H^{-1} \; . \tag{13}$$

We may then take (13) to be a good characterisation of the short-wavelength or Minkowski limit. In this limit, over timescales $\Delta t \equiv 1/\Delta E_{\mathbf{k}} << H^{-1}$, the wave function $\psi_{\mathbf{k}}$ evolves just as it would on Minkowski spacetime. On such timescales, the scale factor a is approximately constant, and the equations (8)–(10) reduce to those of pilot-wave dynamics for an ensemble of nonrelativistic particles of constant mass $m = a^3$ moving in a two-dimensional harmonic oscillator potential of constant angular frequency $\omega = k/a$. From the numerical results for the latter case [18] we may deduce that, in the Minkowski limit, for a decoupled mode $\mathbf{k}$ in a superposition $|\psi_{\mathbf{k}}\rangle \sim |1_{\mathbf{k}}\rangle + |2_{\mathbf{k}}\rangle + |3_{\mathbf{k}}\rangle + \; ...$ of many different states of definite occupation number, the distribution $\rho_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ of the mode amplitudes will relax to equilibrium, $\rho_{\mathbf{k}} \to |\psi_{\mathbf{k}}|^2$ (on a coarse-grained level, again assuming appropriate initial conditions), on a timescale τ given roughly by (12) or

$$\tau \sim \frac{1}{\Delta E_{\mathbf{k}}} = \frac{1}{\Delta n_{\mathbf{k}}} \frac{1}{\omega} \; .$$

3.2 Freezing of the Wave Function for Super-Hubble Modes

In contrast, in the long-wavelength limit,

$$\lambda_{\mathrm{phys}} >> \Delta n_{\mathbf{k}} \cdot H^{-1} \; , \tag{14}$$

we have $\Delta t \equiv 1/\Delta E_{\mathbf{k}} >> H^{-1}$ and the change in the Hamiltonian $\hat{H}(t)$ over timescales H^{-1} may be treated as a sudden perturbation, leading to the conclusion that the wave function $\psi_{\mathbf{k}}$ is approximately static — or 'frozen' — over timescales H^{-1}.

More precisely, let us again consider an evolution over an interval $(t, t + \delta t)$ — but now with δt of order H^{-1}, so that the Hamiltonian $\hat{H}(t)$ changes significantly. We may write $\hat{H}(t+\delta t) = \hat{H}(t) + \delta\hat{H}$, where $\delta\hat{H}$ is comparable to $\hat{H}(t)$. In the limit $\lambda_{\mathrm{phys}} >> \Delta n_{\mathbf{k}} \cdot H^{-1}$, the timescale $\Delta t \equiv 1/\Delta E_{\mathbf{k}}$ associated with the 'unperturbed' Hamiltonian $\hat{H}(t)$ will be large compared to the timescale H^{-1} over which the Hamiltonian changes. We may then treat the change $\delta\hat{H}$ as a sudden perturbation, applied over a timescale that is short compared to the natural timescale of the system. By standard reasoning (for example, ref. [27]),

we deduce that $\psi_{\mathbf{k}}$ hardly changes over the interval $(t, t+\delta t)$, that is, that $\psi_{\mathbf{k}}$ is essentially static over timescales H^{-1}.

Note that the above freezing of the wave function on timescales H^{-1} need not occur for *all* super-Hubble modes, since for any $\lambda_{\mathrm{phys}} > H^{-1}$ the long-wavelength condition (14) will be violated if $\Delta n_{\mathbf{k}}$ is sufficiently large. On the other hand, of course, for any given value of $\Delta n_{\mathbf{k}}$, the condition (14) will be satisfied for sufficiently large λ_{phys} and the wave function will indeed be frozen.

If $\psi_{\mathbf{k}}$ is frozen over timescales H^{-1}, then the equilibrium density $|\psi_{\mathbf{k}}|^2$ is also frozen over timescales H^{-1}. Because the evolution of $|\psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)|^2$ is driven by the de Broglie velocity field $(\dot{q}_{\mathbf{k}1}, \dot{q}_{\mathbf{k}2})$, in accordance with the continuity equation

$$\frac{\partial \left|\psi_{\mathbf{k}}\right|^2}{\partial t} + \frac{\partial}{\partial q_{\mathbf{k}1}}\left(\left|\psi_{\mathbf{k}}\right|^2 \dot{q}_{\mathbf{k}1}\right) + \frac{\partial}{\partial q_{\mathbf{k}2}}\left(\left|\psi_{\mathbf{k}}\right|^2 \dot{q}_{\mathbf{k}2}\right) = 0 \ , \tag{15}$$

we then expect that the trajectories $(q_{\mathbf{k}1}(t), q_{\mathbf{k}2}(t))$ will also be frozen over timescales H^{-1}. (In principle, of course, (15) can have solutions with an essentially static density $|\psi_{\mathbf{k}}|^2$ and a non-negligible velocity field $(\dot{q}_{\mathbf{k}1}, \dot{q}_{\mathbf{k}2})$, but we expect these to occur only in exceptional circumstances. And in any case, because the phase gradient $\partial s_{\mathbf{k}}/\partial q_{\mathbf{k}r}$ is also frozen over timescales H^{-1}, from (9) we see that the velocities $\dot{q}_{\mathbf{k}r}$ become smaller as the scale factor a increases over expansion timescales H^{-1}.) Assuming this to be the case, it then follows that an arbitrary nonequilibrium distribution $\rho_{\mathbf{k}} \neq |\psi_{\mathbf{k}}|^2$, evolving in time according to the same continuity equation (15), will also be frozen over timescales H^{-1}. In other words, at least in this simple case of a decoupled field mode, initial quantum nonequilibrium will be frozen on timescales of order the expansion timescale H^{-1}. (This is reminiscent of the well-known 'freezing' of super-Hubble modes in the theory of cosmological perturbations [28, 29].)

The above reasoning then suggests a mechanism, whereby the rapid expansion of space at early times can suppress the normal process of relaxation to quantum equilibrium, raising the possibility that remnants of early nonequilibrium could have survived to the present day [8, 15]. However, our treatment so far is rather limited. We have considered only a free, decoupled mode in a pure quantum state. It is only expected, and not generally proven, that a frozen $|\psi_{\mathbf{k}}|^2$ will be associated with a family of frozen trajectories. And, perhaps most seriously, while it seems significant to demonstrate nonequilibrium freezing over the (time-dependent) expansion timescale H^{-1}, in a standard — say radiation-dominated — expansion we have $H^{-1} \to 0$ as $t \to 0$, so by itself nonequilibrium freezing over the timescale H^{-1} does not tell us very much about the possible survival of initial nonequilibrium. These limitations will be overcome in the following two sections. We shall first derive a rigorous condition for nonequilibrium freezing, applicable to an arbitrary time interval and to any (generally entangled) pure quantum state of a free field. Then, we shall generalise this condition to mixed states and to interacting fields.

4 Inequality for the Freezing of Quantum Nonequilibrium

To study nonequilibrium freezing over arbitrary time intervals and for arbitrary quantum states, we shall examine the behaviour of the trajectories themselves (instead of the behaviour of their guiding wave functions), thereby obtaining a direct constraint on the evolution of nonequilibrium distributions.

Mathematically, as we saw in section 2, the field system is equivalent to a collection of non-interacting one-dimensional harmonic oscillators with positions $q_{\mathbf{k}r}$ (and with time-dependent masses $m = a^3$ and time-dependent angular frequencies $\omega = k/a$). The Hamiltonian operator is $\hat{H} = \sum_{\mathbf{k}r} \hat{H}_{\mathbf{k}r}$, with

$$\hat{H}_{\mathbf{k}r} = \frac{\hat{\pi}_{\mathbf{k}r}^2}{2a^3} + \frac{1}{2}a^3\omega^2\hat{q}_{\mathbf{k}r}^2 \ .$$

Each $\hat{H}_{\mathbf{k}r}$ has (time-dependent) energy eigenvalues $E_{\mathbf{k}r} = (n_{\mathbf{k}r} + \frac{1}{2})\omega$, where $n_{\mathbf{k}r} = 0, 1, 2, ...$. (Because of the explicit time dependence in the Hamiltonian, the mean energy is of course not conserved: $d\left\langle \hat{H} \right\rangle/dt = \left\langle \partial\hat{H}/\partial t \right\rangle \neq 0$.) For an arbitrary wave functional $\Psi[q_{\mathbf{k}r}, t]$, the de Broglie velocity field is given by (6), and the evolution of an arbitrary ensemble distribution $P[q_{\mathbf{k}r}, t]$ will be driven by this velocity field via the continuity equation (7).

Note that the use of a classical spacetime background must break down in the limit $t \to 0$. The equations defining our model can be trusted only down to some minimum initial time t_i. For example, very optimistically, one might take the 'initial time' to be of order the Planck time, $t_i \sim t_{\mathrm{P}} \sim 10^{-43}$ s.

Now, an initial nonequilibrium distribution $P[q_{\mathbf{k}r}, t_i] \neq |\Psi[q_{\mathbf{k}r}, t_i]|^2$ can in general relax to equilibrium (on a coarse-grained level) only if the trajectories wander sufficiently far over the region of configuration space where $|\Psi|^2$ is concentrated; otherwise, for example, if P were initially small in regions where $|\Psi|^2$ is large, P could remain so, and equilibrium would never be reached. We may then write a simple condition for initial nonequilibrium to be 'frozen', by considering the displacements of the trajectories, and requiring that the (equilibrium) mean magnitude of the displacements be smaller than the width of the wave packet.

Let us write the total configuration of the system as $q(t)$. Note that $\Psi[q, t]$ is in general an entangled function of all the $q_{\mathbf{k}r}$'s. Even so, given the initial distributions $P[q, t_i]$ and $|\Psi[q, t_i]|^2$, one may calculate the corresponding marginals for just one $q_{\mathbf{k}r}$ (for some given $\mathbf{k}r$). If the resulting two marginals are equal or unequal, we may say that we have equilibrium or nonequilibrium respectively, for the given degree of freedom $q_{\mathbf{k}r}$. In this sense, it is clearly possible for some of the $q_{\mathbf{k}r}$'s to be in nonequilibrium while the others are in equilibrium.

Let us now consider the motion $q_{\mathbf{k}r}(t)$ of one degree of freedom, for some given $\mathbf{k}r$, over a time interval $[t_i, t_f]$. An initial point $q_{\mathbf{k}r}(t_i)$ undergoes a final displacement $\delta q_{\mathbf{k}r}(t_f) = \int_{t_i}^{t_f} dt\ \dot{q}_{\mathbf{k}r}(t)$, where the velocity $\dot{q}_{\mathbf{k}r}$ is given by (6).[2]

[2] Note that trajectories in one-dimensional $q_{\mathbf{k}r}$-space do move past each other, being com-

Let $\Delta q_{\mathbf{k}r}(t)$ be the width — with respect to $q_{\mathbf{k}r}$ — of the quantum distribution $|\Psi[q,t]|^2$ at time t. If the whole family of trajectories $q_{\mathbf{k}r}(t)$ (with fixed $\mathbf{k}r$ and arbitrary initial total configurations $q(t_i)$) were such that the magnitude $|\delta q_{\mathbf{k}r}(t_f)|$ of the final displacement were small compared to $\Delta q_{\mathbf{k}r}(t_f)$, then relaxation (with respect to $q_{\mathbf{k}r}$) during the interval $[t_i, t_f]$ would in general be impossible, as the configurations would not move far enough for the two 'fluids' P and $|\Psi|^2$ to be significantly 'stirred' or mixed (with respect to $q_{\mathbf{k}r}$). This is clear because the time evolutions of P and $|\Psi|^2$ are determined by the same continuity equation and the same family of trajectories. For example, if $|\Psi|^2$ is initially spread over an interval $[a,b]$ of $q_{\mathbf{k}r}$-space of length $\sim \Delta q_{\mathbf{k}r}(t_i)$, and if the displacements of all the trajectories during $[t_i, t_f]$ are indeed such that $|\delta q_{\mathbf{k}r}(t_f)| << \Delta q_{\mathbf{k}r}(t_f)$, then $|\Psi|^2$ will essentially remain spread over $[a,b]$ during $[t_i, t_f]$ (with $\Delta q_{\mathbf{k}r}(t_f) \approx \Delta q_{\mathbf{k}r}(t_i)$); while if P is, say, initially confined to the left half of the interval $[a,b]$, it will essentially remain so during $[t_i, t_f]$, and there will be no significant evolution towards equilibrium (for the coordinate $q_{\mathbf{k}r}$).

Thus we might take our condition to be $|\delta q_{\mathbf{k}r}(t_f)| << \Delta q_{\mathbf{k}r}(t_f)$. However, if there were some isolated trajectories for which $|\delta q_{\mathbf{k}r}(t_f)| \sim \Delta q_{\mathbf{k}r}(t_f)$, or even $|\delta q_{\mathbf{k}r}(t_f)| \gtrsim \Delta q_{\mathbf{k}r}(t_f)$, while most trajectories still satisfied $|\delta q_{\mathbf{k}r}(t_f)| << \Delta q_{\mathbf{k}r}(t_f)$ (where 'most' could be defined for example with respect to the Lebesgue measure or with respect to the $|\Psi|^2$-measure), then relaxation would still be impossible in general. Hence we may take the weaker condition

$$\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\mathrm{eq}} << \Delta q_{\mathbf{k}r}(t_f) \ , \tag{16}$$

where $\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\mathrm{eq}}$ is the average of $|\delta q_{\mathbf{k}r}(t_f)|$ over an equilibrium ensemble.

The condition (16) implies that 'most' of the ensemble cannot move by 'much' more than a small fraction of $\Delta q_{\mathbf{k}r}(t_f)$, in the following precise sense. Define $\delta \equiv |\delta q_{\mathbf{k}r}(t_f)| = |q_{\mathbf{k}r}(t_f) - q_{\mathbf{k}r}(t_i)| \geq 0$ (where $\delta = \delta(q_i, t_f)$ is a function of the initial total configuration $q_i \equiv q(t_i)$). From (16), we can write $\langle \delta \rangle_{\mathrm{eq}} < \varepsilon \Delta q_{\mathbf{k}r}(t_f)$ for some $\varepsilon << 1$. We can then show that 'most' values of δ cannot be 'much' bigger than $\varepsilon \Delta q_{\mathbf{k}r}(t_f)$ — where we define 'most' with respect to the equilibrium measure $|\Psi[q_i, t_i]|^2 dq_i$ over the ensemble of initial configurations q_i, and where we define δ to be 'much' bigger than $\varepsilon \Delta q_{\mathbf{k}r}(t_f)$ if $\delta > 2\varepsilon \Delta q_{\mathbf{k}r}(t_f)$. Let R be the set of initial points q_i such that $\delta > \varepsilon \Delta q_{\mathbf{k}r}(t_f) + d$, for some fixed $d > 0$. Such points make up a certain fraction F of the ensemble, that is $F = \int_R dq_i \ |\Psi[q_i, t_i]|^2$. We have a mean

$$\langle \delta \rangle_{\mathrm{eq}} = \int dq_i \ |\Psi[q_i, t_i]|^2 . \delta(q_i, t_f) \ .$$

ponents of higher-dimensional trajectories $q(t)$ (unlike in a strictly one-dimensional system, where the single-valuedness of the velocity field prevents trajectories from crossing).

Since $\delta \geq 0$ for all q_i, we have

$$\begin{aligned}\langle \delta \rangle_{\rm eq} &\geq \int_R dq_i \ |\Psi[q_i, t_i]|^2 . \delta(q_i, t_f) \\ &> \int_R dq_i \ |\Psi[q_i, t_i]|^2 . \left(\varepsilon \Delta q_{\mathbf{k}r}(t_f) + d\right) = F. \left(\varepsilon \Delta q_{\mathbf{k}r}(t_f) + d\right) \ .\end{aligned}$$

Given (16), or $\langle \delta \rangle_{\rm eq} < \varepsilon \Delta q_{\mathbf{k}r}(t_f)$, we then have $\varepsilon \Delta q_{\mathbf{k}r}(t_f) > F. \left(\varepsilon \Delta q_{\mathbf{k}r}(t_f) + d\right)$ and so

$$d < \frac{(1-F)}{F} \varepsilon \Delta q_{\mathbf{k}r}(t_f) \ . \tag{17}$$

Now, if $F > \frac{1}{2}$ (that is, if 'most' initial points yield $\delta > \varepsilon \Delta q_{\mathbf{k}r}(t_f) + d$), then $(1-F)/F < 1$ and so $d < \varepsilon \Delta q_{\mathbf{k}r}(t_f)$. We may then indeed conclude that 'most' of the initial ensemble cannot move by 'much' more than $\varepsilon \Delta q_{\mathbf{k}r}(t_f)$. In this case, even an approximate relaxation cannot (in general) occur.

If (16) is satisfied, then, relaxation will in general be suppressed. Of course, while (16) is a sufficient condition for relaxation suppression, it is not necessary: in principle, the trajectories could even wander over distances larger than $\Delta q_{\mathbf{k}r}(t_f)$ but without a sufficiently complex flow to drive the ensemble towards equilibrium. (As discussed in section 7, it is reasonable to assume that this is unlikely.)

While (16) provides a condition for the freezing of quantum nonequilibrium, in practice it is likely to be more stringent than is necessary. Without attempting to give a rigorous justification, we expect that there will be cases where the weaker condition

$$\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\rm eq} < \Delta q_{\mathbf{k}r}(t_f) \tag{18}$$

suffices to prevent relaxation, at least partially (that is, some significant relaxation towards equilibrium will occur but significant deviations from equilibrium will remain). Generally speaking, we expect that the transition from essentially complete relaxation suppression to essentially full relaxation will take place when the ratio $r \equiv \langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\rm eq} / \Delta q_{\mathbf{k}r}(t_f)$ increases from $r << 1$ to $r \gtrsim 1$, with the critical demarcation line being somewhere in the neigbourhood of $r \sim 1$. We therefore expect that the weaker condition (18) will define (approximately) essentially the whole of the suppression regime, including those cases where significant relaxation towards equilibrium does occur but where significant deviations from equilibrium still remain. (Note that (18) implies that 'most' of the ensemble cannot move by 'much' more than $\Delta q_{\mathbf{k}r}(t_f)$, in the sense given above.)

Pending a more precise treatment, then, here we shall take (18) as our condition for the freezing — or at least partial freezing — of quantum nonequilibrium.

Let us now proceed to draw inferences from (18). Note first that the final displacement $\delta q_{\mathbf{k}r}(t_f)$ has modulus $|\delta q_{\mathbf{k}r}(t_f)| \leq \int_{t_i}^{t_f} dt \ |\dot{q}_{\mathbf{k}r}(t)|$ (where $\int_{t_i}^{t_f} dt \ |\dot{q}_{\mathbf{k}r}(t)|$ is the path length). The equilibrium mean $\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\rm eq}$ then satisfies

$$\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\rm eq} \leq \left\langle \int_{t_i}^{t_f} dt \ |\dot{q}_{\mathbf{k}r}(t)| \right\rangle_{\rm eq} = \int_{t_i}^{t_f} dt \ \langle |\dot{q}_{\mathbf{k}r}(t)| \rangle_{\rm eq} \ , \tag{19}$$

where the equilibrium mean speed $\langle|\dot{q}_{\mathbf{k}r}(t)|\rangle_{\mathrm{eq}}$ at time t is

$$\langle|\dot{q}_{\mathbf{k}r}(t)|\rangle_{\mathrm{eq}} = \int dq\ |\Psi[q,t]|^2 |\dot{q}_{\mathbf{k}r}(q,t)| \tag{20}$$

(the velocity $\dot{q}_{\mathbf{k}r}(q,t)$ being given by (6) as a time-dependent function of the total configuration q).

For the sake of clarity, let us explicitly demonstrate the last equality in (19). The initial equilibrium distribution $|\Psi[q_i,t_i]|^2$ represents an ensemble of initial (total) configurations q_i. From each q_i, the de Broglie velocity field generates a trajectory $q(t)$ (for the whole system), and each such trajectory implies a subsystem trajectory $q_{\mathbf{k}r}(t)$. Thus, at any time t, the subsystem velocity $\dot{q}_{\mathbf{k}r}$ may be regarded as a function of q_i and of t (assuming the wave functional is given). We may then write $\dot{q}_{\mathbf{k}r} = \dot{q}_{\mathbf{k}r}(q_i,t)$ — where of course $\dot{q}_{\mathbf{k}r}(q_i,t)$ and $\dot{q}_{\mathbf{k}r}(q,t)$ here denote two different functions of the first argument. (This notation is strictly speaking ambiguous, but clear from the context.) We then have

$$\langle|\dot{q}_{\mathbf{k}r}(t)|\rangle_{\mathrm{eq}} = \int dq_i\ |\Psi[q_i,t_i]|^2 |\dot{q}_{\mathbf{k}r}(q_i,t)| \tag{21}$$

(with the mean taken over the distribution $|\Psi[q_i,t_i]|^2$ at the fixed *initial* time t_i), so that

$$\begin{aligned}\int_{t_i}^{t_f} dt\ \langle|\dot{q}_{\mathbf{k}r}(t)|\rangle_{\mathrm{eq}} &= \int dq_i\ |\Psi[q_i,t_i]|^2 \left(\int_{t_i}^{t_f} dt\ |\dot{q}_{\mathbf{k}r}(q_i,t)|\right) \\ &= \left\langle \int_{t_i}^{t_f} dt\ |\dot{q}_{\mathbf{k}r}(t)| \right\rangle_{\mathrm{eq}} ,\end{aligned}$$

as used above. (We have shifted notation back and forth, with $\dot{q}_{\mathbf{k}r}(t)$ and $\dot{q}_{\mathbf{k}r}(q_i,t)$ denoting the same thing.)

Using $\langle x\rangle \leq \sqrt{\langle x^2\rangle}$ for any x, we then have

$$\langle|\delta q_{\mathbf{k}r}(t_f)|\rangle_{\mathrm{eq}} \leq \int_{t_i}^{t_f} dt\ \sqrt{\left\langle |\dot{q}_{\mathbf{k}r}(t)|^2 \right\rangle_{\mathrm{eq}}} .$$

Now note that, at any time t,

$$\begin{aligned} a^6 \left\langle|\dot{q}_{\mathbf{k}r}|^2\right\rangle_{\mathrm{eq}} &= \left\langle \left(\frac{\partial S}{\partial q_{\mathbf{k}r}}\right)^2 \right\rangle_{\mathrm{eq}} = \int dq\ |\Psi[q,t]|^2 \left(\frac{\partial S[q,t]}{\partial q_{\mathbf{k}r}}\right)^2 \\ &= \left\langle \hat{\pi}_{\mathbf{k}r}^2 \right\rangle - \int dq\ \left(\frac{\partial|\Psi[q,t]|}{\partial q_{\mathbf{k}r}}\right)^2 \end{aligned} \tag{22}$$

(where $\left\langle\hat{\Omega}\right\rangle$ denotes the usual quantum expectation value for an operator $\hat{\Omega}$). The last equality follows from

$$\left\langle \hat{\pi}_{\mathbf{k}r}^2 \right\rangle = \int dq\ \Psi^* \left(-\frac{\partial^2}{\partial q_{\mathbf{k}r}^2}\right)\Psi = \int dq\ \frac{\partial\Psi^*}{\partial q_{\mathbf{k}r}}\frac{\partial\Psi}{\partial q_{\mathbf{k}r}} ,$$

and from
$$\frac{\partial\Psi^*}{\partial q_{\mathbf{k}r}}\frac{\partial\Psi}{\partial q_{\mathbf{k}r}} = \left(\frac{\partial|\Psi|}{\partial q_{\mathbf{k}r}}\right)^2 + |\Psi|^2\left(\frac{\partial S}{\partial q_{\mathbf{k}r}}\right)^2 \ .$$
Thus, since $(\partial|\Psi|/\partial q_{\mathbf{k}r})^2 \geq 0$, we have
$$a^6\left\langle|\dot{q}_{\mathbf{k}r}|^2\right\rangle_{\mathrm{eq}} \leq \left\langle\hat{\pi}^2_{\mathbf{k}r}\right\rangle \ , \tag{23}$$
and so
$$\left\langle|\delta q_{\mathbf{k}r}(t_f)|\right\rangle_{\mathrm{eq}} \leq \int_{t_i}^{t_f} dt\ \frac{1}{a^3}\sqrt{\left\langle\hat{\pi}^2_{\mathbf{k}r}\right\rangle} \tag{24}$$
(where it is understood that quantities under the integral sign are evaluated at time t).

Since $\left\langle\hat{q}^2_{\mathbf{k}r}\right\rangle > 0$, we also have
$$\left\langle\hat{\pi}^2_{\mathbf{k}r}\right\rangle < 2a^3\left\langle\hat{H}_{\mathbf{k}r}\right\rangle \ , \tag{25}$$
and so
$$\left\langle|\delta q_{\mathbf{k}r}(t_f)|\right\rangle_{\mathrm{eq}} < \int_{t_i}^{t_f} dt\ \frac{1}{a^3}\sqrt{2a^3\left\langle\hat{H}_{\mathbf{k}r}\right\rangle} \ .$$

Introducing the number operator $\hat{n}_{\mathbf{k}r}$, where $\langle\hat{n}_{\mathbf{k}r}\rangle \geq 0$, the mean energy in the mode $\mathbf{k}r$ is
$$\left\langle\hat{H}_{\mathbf{k}r}\right\rangle = (\langle\hat{n}_{\mathbf{k}r}\rangle + \frac{1}{2})\frac{k}{a} \ .$$
We then have
$$\left\langle|\delta q_{\mathbf{k}r}(t_f)|\right\rangle_{\mathrm{eq}} < \int_{t_i}^{t_f} dt\ \frac{1}{a^2}\sqrt{2k(\langle\hat{n}_{\mathbf{k}r}\rangle + 1/2)} \ . \tag{26}$$

The mean $\left\langle|\delta q_{\mathbf{k}r}(t_f)|\right\rangle_{\mathrm{eq}}$ at time t_f is to be compared with the width $\Delta q_{\mathbf{k}r}(t_f)$ (with respect to $q_{\mathbf{k}r}$) of the quantum distribution $|\Psi[q,t_f]|^2$ at time t_f. Using the uncertainty relation $\Delta q_{\mathbf{k}r}\Delta\pi_{\mathbf{k}r} \geq \frac{1}{2}$ and $\Delta\pi_{\mathbf{k}r} \leq \sqrt{\langle\hat{\pi}^2_{\mathbf{k}r}\rangle}$, we have $1/\Delta q_{\mathbf{k}r} \leq 2\sqrt{\langle\hat{\pi}^2_{\mathbf{k}r}\rangle}$. Again using (25) we then have
$$1/\Delta q_{\mathbf{k}r} < 2\sqrt{2a^3\left\langle\hat{H}_{\mathbf{k}r}\right\rangle} = 2a\sqrt{2k(\langle\hat{n}_{\mathbf{k}r}\rangle + 1/2)} \ . \tag{27}$$

Combining the results (26) and (27), we obtain an upper bound for the ratio
$$\frac{\left\langle|\delta q_{\mathbf{k}r}(t_f)|\right\rangle_{\mathrm{eq}}}{\Delta q_{\mathbf{k}r}(t_f)} < 4ka_f\sqrt{\langle\hat{n}_{\mathbf{k}r}\rangle_f + 1/2}\int_{t_i}^{t_f} dt\ \frac{1}{a^2}\sqrt{\langle\hat{n}_{\mathbf{k}r}\rangle + 1/2} \tag{28}$$
(where $a_f \equiv a(t_f)$, and so on). Note that $\langle\hat{n}_{\mathbf{k}r}\rangle$ is in general a function of time t, and that the inequality (28) holds for any arbitrary (in general entangled) state Ψ.

We may now consider the following inequality, that the right-hand side of (28) is less than one, that is

$$\frac{1}{k} > 4a_f \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle_f + 1/2} \int_{t_i}^{t_f} dt \ \frac{1}{a^2} \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle + 1/2} \ . \tag{29}$$

When this 'freezing inequality' is satisfied, $\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\mathrm{eq}} / \Delta q_{\mathbf{k}r}(t_f) < 1$ and initial quantum nonequilibrium will be (at least partially) 'frozen'.

We may also write (29) directly in terms of $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$, yielding

$$4a_f \sqrt{a_f \left\langle \hat{H}_{\mathbf{k}r} \right\rangle_f} \int_{t_i}^{t_f} dt \ \frac{1}{a^2} \sqrt{a \left\langle \hat{H}_{\mathbf{k}r} \right\rangle} < 1 \ . \tag{30}$$

The dependence on the wave number k is of course still present in $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$. Roughly speaking, the freezing inequality (30) requires that the mean energy $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$ in the mode $\mathbf{k}r$ be not too large over the time interval $[t_i, t_f]$ (see below).

5 Generalisations

Before discussing the consequences of the above results, let us first generalise them to more realistic situations. The above derivation of the freezing inequality (29) (or (30)) assumed that the quantum state was pure and that the field was free. The derivation is easily generalised to mixed states and to (finite models of) interacting fields.

With these generalisations in hand, one can then discuss nonequilibrium freezing for a mixed (for example thermal) ensemble of interacting particles, and one can apply the results to realistic models of the early universe.

5.1 Mixed States

In quantum theory, a mixed state is represented by a density operator $\hat{\rho}$, which may be written as a decomposition

$$\hat{\rho} = \sum_{\alpha} p_{\alpha} |\Psi_{\alpha}\rangle\langle\Psi_{\alpha}| \ , \tag{31}$$

with appropriate probability weights p_{α} and pure states $|\Psi_{\alpha}\rangle$. For a scalar field ϕ, the quantum-theoretical distribution for ϕ will be

$$P_{\mathrm{QT}}[\phi, t] = \langle \phi | \hat{\rho}(t) | \phi \rangle = \sum_{\alpha} p_{\alpha} |\Psi_{\alpha}[\phi, t]|^2 \ . \tag{32}$$

The decomposition of $\hat{\rho}$ is generally non-unique, and different decompositions of the same $\hat{\rho}$ are physically equivalent in all respects.

The situation is different in pilot-wave theory. A mixed quantum state is interpreted as a statistical mixture of physically-real pilot waves Ψ_{α}, with probability weights p_{α}, corresponding to a preferred decomposition of $\hat{\rho}$ [30]. For a

given element of the ensemble, the de Broglian velocity of the actual configuration is determined by the actual pilot wave Ψ_α. A different decomposition of $\hat{\rho}$ would generally yield different velocities, and so be physically distinct at the fundamental level. (Note that, in quantum nonequilibrium, the velocities and trajectories for single systems can be measured without necessarily disturbing the wave functions [11, 14], enabling the preferred decomposition to be detected. The operational equivalence of different decompositions of $\hat{\rho}$ is a peculiarity of the quantum equilibrium state; see ref. [13].)

Now, given such a preferred decomposition, for each pure subensemble with wave functional $\Psi_\alpha[\phi, t]$ — taking the system to consist of a scalar field ϕ — we may define a distribution $P_\alpha[\phi, t]$ (generally $\neq |\Psi_\alpha[\phi, t]|^2$) and an associated H-function

$$H_\alpha = \int D\phi \; P_\alpha \ln(P_\alpha / |\Psi_\alpha|^2)$$

(for some appropriate measure $D\phi$). The whole ensemble has a distribution

$$P[\phi, t] = \sum_\alpha p_\alpha P_\alpha[\phi, t] \; , \tag{33}$$

and the mean H-function

$$H = \sum_\alpha p_\alpha H_\alpha \tag{34}$$

obeys a coarse-graining H-theorem (for a closed system with constant p_α) [13]. The equilibrium minimum $H = 0$ (which may be approached in a coarse-grained sense) corresponds to $H_\alpha = 0$ and $P_\alpha = |\Psi_\alpha|^2$ for every α, so that (33) reduces to (32).

Thus, we may discuss relaxation for a mixed state in terms of relaxation for its component pure subensembles. We may then consider the freezing inequality (29) (or (30)) for each pure subensemble separately. Clearly, the inequality might hold for some subensembles and not for others (or for all of them, or none).

If the (quantum) mean occupation number for state $|\Psi_\alpha\rangle$ is $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha \equiv \langle \Psi_\alpha | \, \hat{n}_{\mathbf{k}r} \, | \Psi_\alpha \rangle$, then for a mixed state (31) the overall mean occupation number will be

$$\overline{\langle \hat{n}_{\mathbf{k}r} \rangle} = \sum_\alpha p_\alpha \left\langle \hat{n}_{\mathbf{k}r} \right\rangle_\alpha \; .$$

For example, for a thermal ensemble with temperature T, we will have the Planck distribution

$$\overline{\langle \hat{n}_{\mathbf{k}r} \rangle}_{\mathrm{P}} = \frac{1}{e^{\hbar\omega / k_{\mathrm{B}} T} - 1} \; .$$

In general, $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$ for a pure subensemble will differ from $\overline{\langle \hat{n}_{\mathbf{k}r} \rangle}$, and the total ensemble will contain a range of different values for $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$. Initial quantum nonequilibrium will be frozen for the pure subensemble with wave functional Ψ_α, if the corresponding quantity $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$ satisfies the freezing inequality (29) (with $\langle \hat{n}_{\mathbf{k}r} \rangle$ replaced by $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$).

To investigate which (if any) pure subensembles will satisfy the freezing inequality (29), we need to know the quantities $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$ as functions of time, that is, we need to know which pure states $|\Psi_\alpha\rangle$ are present in the total ensemble. Despite the operational equivalence of different decompositions of $\hat{\rho}$ in quantum theory, it has been argued that, in the case of thermal (canonical) ensembles, there is a natural probability measure on the space of normalised wave functions, the 'Gaussian adjusted projected measure', which is unique for each $\hat{\rho}$, and which may be used to define a preferred decomposition [31]. This proposal has been applied to the case of an ideal gas (though described in terms of particle theory rather than field theory) [32]. For our purposes, we would need to apply the preferred measure to a thermal ensemble of wave functionals in field theory on expanding space, and use the results to deduce which (if any) subensembles of finite measure satisfy the freezing inequality (29) (or (30)). We do not attempt such a calculation here, but it should be clear that the problem is well-defined.

If certain pure subensembles — with labels α in some set S — are predicted to be frozen, then (assuming initial nonequilibrium) the total ensemble distribution of ϕ will take the form

$$P[\phi, t] = \sum_{\alpha \in S} p_\alpha P_\alpha[\phi, t] + \sum_{\alpha \notin S} p_\alpha |\Psi_\alpha[\phi, t]|^2 \ ,$$

where $P_\alpha \neq |\Psi_\alpha|^2$ (for $\alpha \in S$), and $P[\phi, t]$ will generally differ from the equilibrium result (32).

The physics of nonequilibrium mixed states needs further development. In particular, one should explore how measurements could probe the nonequilibrium physics particular to a specific pure subensemble (noting again that, unlike in quantum theory, in nonequilibrium pilot-wave theory it is operationally meaningful to speak of the physics of component pure subensembles). However, the above suffices for the purposes of this paper.

5.2 Interacting Fields

Our derivation in section 4 of the freezing inequality (29) assumed that the field ϕ was free. The derivation is easily generalised to interacting fields, at least if one considers finite models with an appropriate high-frequency cutoff (so that divergences may be ignored).

Let the scalar field ϕ interact with other fields, denoted collectively by Φ. (These other fields need not be scalars.) We have a total Hamiltonian

$$\hat{H}_{\text{total}} = \hat{H} + \hat{H}_\Phi + \hat{H}_{\text{I}} \ ,$$

where $\hat{H}$ and $\hat{H}_\Phi$ are respectively the free Hamiltonians for ϕ and Φ, while $\hat{H}_{\text{I}}$ is the interaction Hamiltonian.

We may still of course write ϕ in terms of its Fourier components $\phi_{\mathbf{k}r}$, and the free Hamiltonian $\hat{H}$ still decomposes into a sum $\hat{H} = \sum_{\mathbf{k}r} \hat{H}_{\mathbf{k}r}$, with $\hat{H}_{\mathbf{k}r} = (\hat{n}_{\mathbf{k}r} + \frac{1}{2})\frac{k}{a}$, exactly as before. Equation (22) still holds (for a pure subensemble with wave functional Ψ, and where the total configuration q now includes Φ as

well as ϕ). So we still have the inequality (23). The other inequalities — such as (25) and $\Delta q_{\mathbf{k}r} \Delta \pi_{\mathbf{k}r} \geq \frac{1}{2}$ — are also valid as in the case of a free field. We therefore arrive again at the upper bound (28) and the freezing inequalities (29) and (30).

The only difference from the free case is in the time evolution of $\langle \hat{n}_{\mathbf{k}r} \rangle$ (or of $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$), which now involves contributions from $\hat{H}_I$:

$$\frac{d \langle \hat{n}_{\mathbf{k}r} \rangle}{dt} = \left\langle \frac{\partial \hat{n}_{\mathbf{k}r}}{\partial t} \right\rangle - i \left\langle [\hat{n}_{\mathbf{k}r}, \hat{H}_I] \right\rangle \ .$$

The calculation of $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ as a function of time t will then be more complicated than in the free case, where only the first term appears on the right hand side. (The evolution of $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ in the free case is studied in section 8.)

6 General Implications of the Freezing Inequality

Quite generally, then, even for an interacting field in a mixed state, we may conclude that relaxation will be suppressed — that is, nonequilibrium will be frozen — for modes whose (time-dependent) mean occupation number $\langle \hat{n}_{\mathbf{k}r} \rangle$ satisfies the inequality (29).

For a given time evolution, defined by $a(t)$ and $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ (for all $\mathbf{k}r$) on $[t_i, t_f]$, it is of course possible that (29) will not be satisfied for any value of k, and that all modes relax (at least approximately) towards equilibrium during the interval $[t_i, t_f]$. On the other hand if, for a given time evolution, (29) is satisfied only for certain values of k, then we can predict that significant deviations from quantum equilibrium are to be expected only for those particular values of k.

We emphasise that, for each mode, whether or not the inequality (29) is satisfied depends on the history of the expansion and on the time evolution of the quantum state of the field.

For a radiation-dominated expansion on $[t_i, t_f]$, with $a(t) = a_f (t/t_f)^{1/2}$, we may make a general statement about the kind of modes that can satisfy (29): the physical wavelength $\lambda_{\mathrm{phys}}(t_f) = a_f(2\pi/k)$ at time t_f must be larger than the Hubble radius H_f^{-1} at time t_f (assuming that $t_f \gtrsim (1.17)t_i$).

This is easily shown for any quantum state. Since $\langle \hat{n}_{\mathbf{k}r} \rangle \geq 0$, the inequality (29) (assuming it to hold) implies that

$$\frac{1}{k} > 2a_f \int_{t_i}^{t_f} dt \ \frac{1}{a^2} = \frac{2t_f}{a_f} \ln(t_f/t_i) \ ,$$

or

$$\lambda_{\mathrm{phys}}(t_f) > 2\pi H_f^{-1} \ln(t_f/t_i) \ , \tag{35}$$

where $H_f^{-1} = 2t_f$ and where the right-hand side is indeed larger than H_f^{-1} if $t_f > t_i \exp(1/2\pi) \simeq (1.17)t_i$. (This is of course not to suggest that the freezing

inequality is satisfied for all super-Hubble modes: rather, if the inequality is satisfied, then the corresponding modes must be super-Hubble.)

Note that, in any reasonable application of this result, the factor $\ln(t_f/t_i)$ will not be large. For example, taking $t_i \sim t_{\rm P} \sim 10^{-43}$ s, for $t_f \sim 10^{-35}$ s (the time at which inflation begins in some models [28]) we have $\ln(t_f/t_i) \sim \ln 10^8 \sim 20$, while even for $t_f \sim 1$ s (the time of neutrino decoupling) we have $\ln(t_f/t_i) \sim \ln 10^{43} \sim 10^2$. The factor $2\pi \ln(t_f/t_i)$ is then likely to be at most of order $10^2 - 10^3$, in which case the minimal value of $\lambda_{\rm phys}(t_f)$ for nonequilibrium field modes will be at most two or three orders of magnitude larger than the Hubble radius H_f^{-1}.

On the other hand, again for a radiation-dominated expansion, the true lower bound on $\lambda_{\rm phys}(t_f)$ (set by (29)) will be much larger than $2\pi H_f^{-1} \ln(t_f/t_i)$ if $\langle \hat{n}_{\mathbf{k}r} \rangle_t >> 1$ during the period $[t_i, t_f]$, as is clear from (29).

Thus, de Broglie-Bohm theory (with the assumption of early quantum nonequilibrium at some initial time t_i) predicts that residual or 'frozen' nonequilibrium will exist at later times $t_f > t_i$ for modes satisfying the inequality (29), where for a radiation-dominated expansion the physical wavelength $\lambda_{\rm phys}(t_f)$ of nonequilibrium modes at time t_f must be bigger than $2\pi H_f^{-1} \ln(t_f/t_i)$ (at least).

If we take the freezing inequality in the form (30), we see that, roughly speaking, it entails an upper bound on the mean energy $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$ per mode over time. More precisely if, for example, $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle \geq \left\langle \hat{H}_{\mathbf{k}r} \right\rangle_{\rm min}$ throughout $[t_i, t_f]$, then (30) implies that

$$\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_{\rm min} < \frac{1}{4a_f^{3/2} \int_{t_i}^{t_f} dt \; a^{-3/2}} \; . \tag{36}$$

For a radiation-dominated expansion, and assuming $t_f/t_i >> 1$, we then have (inserting $\hbar$)

$$\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_{\rm min} < \frac{\hbar}{16 t_f} = \frac{\hbar}{8 H_f^{-1}} \tag{37}$$

(where, dimensionally speaking, $H_f^{-1} = 2t_f$ is the Hubble time and cH_f^{-1} is the Hubble radius).

Finally, we note that *violation* of the freezing inequality (29) in the infra-red limit $k \to 0$ requires that $\langle \hat{n}_{\mathbf{k}r} \rangle$ be divergent as $k \to 0$. Alternatively, for (30) to be violated as $k \to 0$, the mean energy per mode $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$ must remain finite as $k \to 0$.

7 Relaxation for Modes Violating the Freezing Inequality

We have shown that, for modes satisfying (29), relaxation will be suppressed over the time interval $[t_i, t_f]$. For a radiation-dominated expansion we know from (35) that such modes, if they exist, must have super-Hubble wavelengths.

Further, as discussed in section 3, we know from previous studies that relaxation is likely to occur in the short-wavelength (Minkowski) limit. What can we say about modes that violate the freezing inequality (29) without approaching the Minkowski limit?

Our derivation of the upper bound (28) made use of several general inequalities (such as $\langle \hat{\pi}^2_{\mathbf{k}r} \rangle \leq 2a^3 \left\langle \hat{H}_{\mathbf{k}r} \right\rangle$). For a large class of quantum states, these general inequalities could be replaced by approximate equalities, to be used as rough, order-of-magnitude estimates (for example, $\langle \hat{\pi}^2_{\mathbf{k}r} \rangle \sim 2a^3 \left\langle \hat{H}_{\mathbf{k}r} \right\rangle$). For such states, then, we have an estimated ratio

$$\frac{\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\mathrm{eq}}}{\Delta q_{\mathbf{k}r}(t_f)} \sim 4ka_f \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle_f + 1/2} \int_{t_i}^{t_f} dt\ \frac{1}{a^2} \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle + 1/2}\ .$$

It then follows that if (instead of (29))

$$\frac{1}{k} \lesssim 4a_f \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle_f + 1/2} \int_{t_i}^{t_f} dt\ \frac{1}{a^2} \sqrt{\langle \hat{n}_{\mathbf{k}r} \rangle + 1/2}\ , \tag{38}$$

or equally if (instead of (30))

$$4a_f \sqrt{a_f \left\langle \hat{H}_{\mathbf{k}r} \right\rangle_f} \int_{t_i}^{t_f} dt\ \frac{1}{a^2} \sqrt{a \left\langle \hat{H}_{\mathbf{k}r} \right\rangle} \gtrsim 1\ , \tag{39}$$

then

$$\frac{\langle |\delta q_{\mathbf{k}r}(t_f)| \rangle_{\mathrm{eq}}}{\Delta q_{\mathbf{k}r}(t_f)} \gtrsim 1\ . \tag{40}$$

From this we may reasonably deduce that relaxation, or at least significant relaxation, is likely to occur (except of course for special states with very simple velocity fields).

Unlike our proof of relaxation suppression for modes satisfying (29), this is not a rigorous result. (It is roughly analogous to saying, in classical kinetic theory, that significant relaxation to thermal equilibrium is likely to occur, over timescales of order the mean free time, if the mean magnitude of momentum transferred in molecular collisions is comparable to the width of the equilibrium momentum distribution.) To delineate the precise behaviour in this region requires further study, perhaps through numerical simulations.

To avoid potential misunderstandings, we should emphasise that relaxation might of course be suppressed for special quantum states violating the freezing inequality (29) (in particular, states with an especially simple de Broglie velocity field). However, one should bear in mind that we are concerned with the evolution of quantum nonequilibrium in our actual universe, which is known to have had a complex and violent past history. Thus, for example, in a standard radiation-dominated phase, special states with no entanglement at any time are of no interest: we are concerned with states that are likely to have actually occurred. In seeking a general criterion for the freezing of early nonequilibrium, it is then of no use to point to special quantum states exhibiting particularly

simple velocity fields.[3] In contrast, the freezing inequality (29) is a natural constraint on quantum states in general, providing a realistic pointer to where nonequilibrium might be found in our actual universe. And violation of (29) is, as we have argued in this section, likely to imply relaxation or at least significant relaxation.

8 Time Evolution of $\langle \hat{n}_{\mathbf{k}r} \rangle$ for a Free Field

For a given expansion history $a = a(t)$ on $[t_i, t_f]$, the freezing inequality (29) depends on the time evolution of $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ on $[t_i, t_f]$ (or, (30) depends on $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_t$). To make precise predictions, then, we require a specific cosmological model, and an explicit expression for $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ as a function of time t. We leave such detailed studies for future work. Here, we give a method for calculating $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ (and $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_t$) for an arbitrary pure quantum state. This method might prove useful.

The mean energy

$$W_{\mathbf{k}r} \equiv \left\langle \hat{H}_{\mathbf{k}r} \right\rangle = (\langle \hat{n}_{\mathbf{k}r} \rangle + 1/2)(k/a)$$

in the mode $\mathbf{k}r$ evolves in time according to $dW_{\mathbf{k}r}/dt = \left\langle \partial \hat{H}_{\mathbf{k}r}/\partial t \right\rangle$, which implies (using $\dot{a} = Ha$)

$$\frac{dW_{\mathbf{k}r}}{dt} = -3HW_{\mathbf{k}r} + 4HU_{\mathbf{k}r} \ , \tag{41}$$

where $U_{\mathbf{k}r} \equiv \left\langle \frac{1}{2} a^3 \omega^2 \hat{q}_{\mathbf{k}r}^2 \right\rangle$ is the mean potential energy. (For an interacting field, as discussed in section 5.2, $dW_{\mathbf{k}r}/dt$ would contain additional terms from $-i \left\langle [\hat{H}_{\mathbf{k}r}, \hat{H}_I] \right\rangle$.) The rate of change of $\langle \hat{n}_{\mathbf{k}r} \rangle = (a/k) W_{\mathbf{k}r} - 1/2$ is then given by

$$\frac{d \langle \hat{n}_{\mathbf{k}r} \rangle}{dt} = -2(Ha/k)(K_{\mathbf{k}r} - U_{\mathbf{k}r}) \ , \tag{42}$$

where $K_{\mathbf{k}r} \equiv \left\langle \hat{\pi}_{\mathbf{k}r}^2 / 2a^3 \right\rangle$ is the mean kinetic energy.

To solve for $W_{\mathbf{k}r}(t) = K_{\mathbf{k}r}(t) + U_{\mathbf{k}r}(t)$, and hence the required function $\langle \hat{n}_{\mathbf{k}r} \rangle_t$, one may write first-order (linear) differential equations for $K_{\mathbf{k}r}$, $U_{\mathbf{k}r}$ and for the quantity $\chi_{\mathbf{k}r} \equiv \frac{1}{2} \langle \hat{q}_{\mathbf{k}r} \hat{\pi}_{\mathbf{k}r} + \hat{\pi}_{\mathbf{k}r} \hat{q}_{\mathbf{k}r} \rangle$. Using $d \left\langle \hat{\Omega} \right\rangle / dt = -i \left\langle [\hat{\Omega}, \hat{H}] \right\rangle + \left\langle \partial \hat{\Omega} / \partial t \right\rangle$, it is readily shown that

$$\frac{dK_{\mathbf{k}r}}{dt} = -3HK_{\mathbf{k}r} - \omega^2 \chi_{\mathbf{k}r} \ , \quad \frac{dU_{\mathbf{k}r}}{dt} = HU_{\mathbf{k}r} + \omega^2 \chi_{\mathbf{k}r} \ , \quad \frac{d\chi_{\mathbf{k}r}}{dt} = 2(K_{\mathbf{k}r} - U_{\mathbf{k}r}) \ . \tag{43}$$

[3] A notable exception is the inflationary vacuum, which is in fact an example of a state that is non-entangled (across modes), with a very simple velocity field, and which is widely believed to have existed in the past; see section 10.1.

If $H = \dot{a}/a$ and $\omega = k/a$ are known functions of time, then given values of $K_{\mathbf{k}r}$, $U_{\mathbf{k}r}$, $\chi_{\mathbf{k}r}$ at any one time (say t_i or t_f) — where these values are determined by the wave functional Ψ at that time[4] — the equations (43) determine $K_{\mathbf{k}r}$, $U_{\mathbf{k}r}$, $\chi_{\mathbf{k}r}$ at all times, yielding $W_{\mathbf{k}r}(t) = K_{\mathbf{k}r}(t) + U_{\mathbf{k}r}(t)$ as well as the required function $\langle \hat{n}_{\mathbf{k}r} \rangle_t = a(t) W_{\mathbf{k}r}(t)/k - 1/2$.

Introducing the vector $X = (K_{\mathbf{k}r}, U_{\mathbf{k}r}, \chi_{\mathbf{k}r})^{\mathrm{T}}$, the equations (43) take the form $dX/dt = AX$, where A is the time-dependent matrix

$$A = \begin{pmatrix} -3\dot{a}/a & 0 & -k^2/a^2 \\ 0 & \dot{a}/a & k^2/a^2 \\ 2 & -2 & 0 \end{pmatrix} . \tag{44}$$

For interesting forms of a, such as $a \propto t^{1/2}$, it seems likely that these equations will have to be solved numerically.

It would be interesting to study this system of equations, and to establish the conditions under which solutions for $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ (or $W_{\mathbf{k}r}(t) = \left\langle \hat{H}_{\mathbf{k}r} \right\rangle_t$) satisfy the freezing inequality (29) (or (30)). We leave this for future work.

9 Approximate Solutions for $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ Satisfying the Freezing Inequality

However, it is important to show first of all that solutions for $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ satisfying (29) can exist for some values of k. Here, we construct approximate solutions of (43) valid in the long-wavelength limit $k \to 0$, that satisfy (29) for appropriate initial conditions and time intervals. The conditions of validity are probably too restrictive for useful application to realistic cosmological scenarios, and we give these solutions here only to show that solutions satisfying (29) are indeed possible.

We consider a radiation-dominated expansion, for which $a \propto t^{1/2}$ and $H = 1/2t$. Dropping the indices $\mathbf{k}r$, we find approximate solutions to (43) satisfying (for appropriate values of k)

$$\omega^2 |\chi| << HK,\ HU \tag{45}$$

(where $K,\ U$ are non-negative), or

$$\frac{k^2 t_i}{a_i^2} |\chi| << K,\ U \tag{46}$$

(where $t_i/a_i^2 = t_f/a_f^2 = t/a^2$). We then have the simple solutions

$$K = K_i (t/t_i)^{-3/2} , \qquad U = U_i (t/t_i)^{1/2} \tag{47}$$

[4] Of course, initial values for $K_{\mathbf{k}r}$, $U_{\mathbf{k}r}$, $\chi_{\mathbf{k}r}$ cannot be chosen completely arbitrarily. They are subject to constraints, such as $a^3 K + U/(ak^2) + \chi = \frac{1}{2} \left\langle (\hat{\pi} + \hat{q})^2 \right\rangle \geq 0$ (or $ak^2 K + U/(a^3) + \omega^2 \chi \geq 0$).

(that is, $K \propto 1/a^3$ and $U \propto a$), and

$$\chi = \chi_i + 4K_i t_i \left(1 - (t/t_i)^{-1/2}\right) + \frac{4}{3} U_i t_i \left(1 - (t/t_i)^{3/2}\right) \ . \tag{48}$$

Note that, for these solutions, the quantities $\left\langle \hat{q}^2_{\mathbf{k}r} \right\rangle = 2U_{\mathbf{k}r}/(k^2 a)$ and $\left\langle \hat{\pi}^2_{\mathbf{k}r} \right\rangle = 2a^3 K_{\mathbf{k}r}$ are time independent.

We need to show the consistency of the solutions (47) and (48) with the assumed approximation (46). This may be done if k is appropriately small. Specifically, writing

$$\chi = \chi_i + 4K_i t_i + \frac{4}{3} U_i t_i - 4Kt - \frac{4}{3} Ut \ ,$$

we have (since Kt and Ut respectively decrease and increase with time)

$$|\chi| \leq |\chi_i| + 8K_i t_i + \frac{8}{3} U_f t_f \equiv D \ .$$

If we assume that

$$k^2 << \frac{a_i^2}{t_i^2} \frac{t_i}{D} \min \{K_f, U_i\} \tag{49}$$

(where $(t_i/D) \min \{K_f, U_i\}$ is dimensionless), we then have

$$\frac{k^2 t_i}{a_i^2} |\chi| \leq \frac{k^2 t_i}{a_i^2} D << \min \{K_f, U_i\} \leq K, \ U$$

(since K and U respectively decrease and increase), and so the approximation condition (46) is indeed satisfied.

For k satisfying (49), we then have the approximate solutions (47) for K and U. We wish to show explicitly that, for these solutions, there are values of k that satisfy the freezing inequality (29) (or (30)).

To show this, for simplicity we first choose initial conditions with $K_i << U_i$. Since K decreases with time, we then have $\min \{K_f, U_i\} = K_f$ and (from (49)) the solutions (47) are valid if

$$k^2 << \frac{a_i^2}{t_i} \frac{K_f}{D} \tag{50}$$

(where $a_i^2/t_i = a_f^2/t_f = a^2/t$). Further, since K decreases and U increases with time, $K_i << U_i$ implies that $K << U$ for all $t \geq t_i$. Thus we have $\left\langle \hat{H} \right\rangle_t \approx U(t)$ (where we continue to suppress the indices $\mathbf{k}r$), or (using (47))

$$\left\langle \hat{H} \right\rangle_t \approx U_i (t/t_i)^{1/2} \ . \tag{51}$$

Inserting this into the freezing inequality (30), and using $a = a_f (t/t_f)^{1/2}$ and $H_f^{-1} = 2t_f$, and taking $t_i/t_f << 1$, we obtain

$$U_i < \frac{1}{4} \frac{a_i}{a_f} \frac{1}{H_f^{-1}} \ . \tag{52}$$

Since $\left\langle \hat{H} \right\rangle_i \approx U_i$ we have (restoring indices $\mathbf{k}r$) the freezing inequality

$$\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i < \frac{1}{4} \frac{a_i}{a_f} \frac{1}{H_f^{-1}} = \frac{1}{4} \left(\frac{a_i}{a_f} \right)^3 \frac{1}{H_i^{-1}} \ . \tag{53}$$

(Note that, since for the above solution $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle$ increases with time, the general result (37) also applies, with $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i = \left\langle \hat{H}_{\mathbf{k}r} \right\rangle_{\text{min}} < 1/8H_f^{-1}$. This is consistent with (53), since we have assumed $t_i/t_f << 1$ which implies $a_i/a_f << 1$.)

Thus, for a given mode $\mathbf{k}r$ satisfying (50), if $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i$ is sufficiently small (satisfying (53)), then relaxation will be suppressed and initial nonequilibrium (if it exists) will be frozen. And it is indeed always possible to choose $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i$ so as to satisfy (53), provided k is sufficiently small. For the only general constraint on $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i$ is

$$\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i = (\langle \hat{n}_{\mathbf{k}r} \rangle_i + \frac{1}{2}) \frac{k}{a_i} \geq \frac{1}{2} \frac{k}{a_i} \ ,$$

so it is possible to satisfy (53) if

$$k < \frac{1}{2} \frac{a_i}{a_f} \frac{a_i}{H_f^{-1}}$$

or

$$\lambda_{\text{phys}} = a_f \lambda > 4\pi \left(\frac{a_f}{a_i} \right)^2 H_f^{-1} >> H_f^{-1} \ .$$

If instead we choose initial conditions with $K_i >> U_i$, we will have $K >> U$ only for as long as t_i/t_f is not much smaller than 1. Over this limited time, we have

$$\left\langle \hat{H} \right\rangle_t \approx K(t) = K_i (t/t_i)^{-3/2} = K_f (t/t_f)^{-3/2} \ . \tag{54}$$

Since $K_f >> U_f > U_i$, we now have $\min \{K_f, U_i\} = U_i$ and the solution is valid if

$$k^2 << \frac{a_i^2}{t_i} \frac{U_i}{D} \ . \tag{55}$$

Inserting (54) into the freezing inequality (30), and assuming that t_i/t_f is small compared to 1 (but not so small as to invalidate the approximation $K >> U$), we obtain the freezing inequality

$$\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i \approx K_i < \frac{1}{4} \frac{1}{H_i^{-1}} \ . \tag{56}$$

Again using $\left\langle \hat{H}_{\mathbf{k}r} \right\rangle_i \geq (1/2)(k/a_i)$, we now find that it is possible to satisfy (56) if

$$k < \frac{1}{2} \frac{a_i}{H_i^{-1}} = \frac{1}{2} \left(\frac{a_f}{a_i} \right)^2 \frac{a_i}{H_f^{-1}}$$

or

$$\lambda_{\rm phys} = a_f \lambda > 4\pi \frac{a_i}{a_f} H_f^{-1} \ .$$

By assumption, a_i/a_f is not much smaller than 1, so we still have $\lambda_{\rm phys} \gtrsim H_f^{-1}$. (In any case, it follows from (55) that the solution is valid only if $a_f\lambda > H_f^{-1}$. For we have

$$D \geq 8K_i t_i + \frac{8}{3} U_f t_f > 8K_i t_i + \frac{8}{3} U_i t_i \approx 8K_i t_i \ ,$$

so that (55) gives

$$k^2 << \frac{a_i^2}{t_i} \frac{U_i}{D} \lesssim \frac{a_i^2}{t_i^2} \frac{U_i}{8K_i} << \left(\frac{a_i}{t_i} \right)^2$$

or $a_f\lambda >> (a_i/a_f) H_f^{-1}$. Since a_i/a_f is not much smaller than 1, we indeed have $a_f\lambda > H_f^{-1}$.)

It is therefore certainly possible to have solutions for $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ with nonequilibrium freezing for long-wavelength modes, $a_f\lambda > H_f^{-1}$ (or $a_f\lambda >> H_f^{-1}$).

10 Possible Consequences of Early Nonequilibrium Freezing

The freezing inequality (29) (or (30)) makes it possible, for the first time, to make quantitative predictions for nonequilibrium deviations from quantum theory, if we are given a specific cosmological model. The potential consequences are many, and much remains to be done to develop them. Here, we restrict ourselves to a preliminary sketch of some possible nonequilibrium effects, in particular: corrections to inflationary predictions for the CMB, non-inflationary super-Hubble field correlations, and relic nonequilibrium particles. We hope to develop further details elsewhere, in the context of specific (and realistic) cosmological models.

As we saw in section 6, for a radiation-dominated expansion (29) implies the general lower bound (35) on the physical wavelength $\lambda_{\rm phys}(t_f)$ — the wavelength of what might be termed 'relic nonequilibrium field modes' — at the final time t_f. In terms of the ambient temperature T, where $T \propto 1/a \propto t^{-1/2}$, the lower bound may be written as

$$\lambda_{\rm phys}(t_f) > 4\pi H_f^{-1} \ln(T_i/T_f) \ . \qquad (57)$$

As we have discussed, this lower bound will in practice be not more than two or three orders of magnitude larger than the Hubble radius H_f^{-1} at time t_f.

Note that, to satisfy the freezing inequality, the bound (57) is a necessary but not sufficient condition. A detailed understanding of where nonequilibrium freezing can occur requires, as discussed in section 5.1, a calculation of the time evolution of the mean occupation numbers $\langle \hat{n}_{\mathbf{k}r} \rangle_\alpha$ for the pure subensembles (with wave functionals Ψ_α) contained in the early mixed state, to find out which

— if any — of these subensembles satisfy (29). This is a matter for future work. Here, we consider only the necessary condition (57), which provides a pointer to where residual nonequilibrium *could* be found (pending the said more complete analysis). In particular, (57) suggests that one should look for nonequilibrium above a specific critical wavelength.

10.1 Corrections to Inflationary Predictions for the CMB

In inflationary cosmology, the universe undergoes a period of exponential expansion, $a(t) \propto e^{Ht}$, driven by the energy density of an approximately homogeneous scalar or inflaton field ϕ, where quantum fluctuations in ϕ seed the primordial curvature perturbations that are later imprinted as temperature anisotropies in the CMB [29].

To a first approximation, inflation predicts that modes of the inflaton field will have a quantum variance

$$\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}} = \frac{V}{2(2\pi)^3} \frac{H^2}{k^3} \tag{58}$$

and a scale-invariant power spectrum

$$\mathcal{P}_{\phi}^{\mathrm{QT}}(k) \equiv \frac{4\pi k^3}{V} \left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}} = \frac{H^2}{4\pi^2} \ , \tag{59}$$

where $\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}}$ is obtained from the Bunch-Davies vacuum in de Sitter space, for $\lambda_{\mathrm{phys}} >> H^{-1}$. In the slow-roll limit ($\dot{H} \to 0$), this results in a scale-invariant spectrum, $\mathcal{P}_{\mathcal{R}}^{\mathrm{QT}}(k) = \text{const.}$, for the primordial curvature perturbation $\mathcal{R}_{\mathbf{k}}$, in approximate agreement with what is observed in the CMB [33].

Now, quantum nonequilibrium in the early Bunch-Davies vacuum generally implies deviations from (58). It has been shown [15, 20] that if (microscopic) quantum nonequilibrium exists at the onset of inflation, then instead of relaxing it will be preserved during the inflationary phase, and furthermore it will be transferred to macroscopic lengthscales by the expansion of physical wavelengths $\lambda_{\mathrm{phys}} \propto a(t) \propto e^{Ht}$. Specifically, for each mode $\mathbf{k}$, explicit calculation shows that the width of the evolving nonequilibrium distribution remains in a constant ratio with the width of the equilibrium distribution. (This is essentially because the vacuum state has the special property of being non-entangled across modes, so that the de Broglie-Bohm trajectories decompose into independent one-dimensional motions. See ref. [20].) If we write the nonequilibrium variance as

$$\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle = \left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}} \xi(k) \tag{60}$$

(where equilibrium corresponds of course to $\xi(k) = 1$ for all k), the power spectrum for $\mathcal{R}_{\mathbf{k}}$ is then just the quantum result multiplied by the 'nonequilibrium factor' $\xi(k)$: that is, $\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\mathrm{QT}}(k)\xi(k)$.

Thus, quantum nonequilibrium at the beginning of inflation will generally break the scale invariance of $\mathcal{P}_{\mathcal{R}}(k)$. As discussed in detail elsewhere [20], measurements of the angular power spectrum for the CMB may be used (in the context of inflation) to set bounds on $\xi(k)$.

Given these results, the next step is to try to predict some features of the function $\xi(k)$. This requires a constraint on the form of nonequilibrium at the onset of inflation.

One possible strategy is to consider a pre-inflationary era, and to derive constraints on residual nonequilibrium from that era. If we take the pre-inflationary era to be radiation-dominated ($a \propto t^{1/2}$), the lower bound (57) shows that nonequilibrium (for whatever fields may be present in that era) can survive only for sufficiently large, super-Hubble wavelengths. Since $\lambda_{\mathrm{phys}} \propto t^{1/2}$ and $H^{-1} \propto t$, at sufficiently early times all physical wavelengths will in fact be super-Hubble ($\lambda_{\mathrm{phys}} > H^{-1}$), raising the possibility of nonequilibrium freezing for the corresponding modes (if the freezing inequality (29) is satisfied). During the subsequent inflationary phase, H^{-1} is (approximately) constant, and relevant cosmological fluctuations originate from inside H^{-1}. Some of these fluctuating modes could be out of equilibrium only if they evolved from modes that were outside the Hubble radius in the pre-inflationary phase.

Thus, in order to obtain nonequilibrium corrections to inflationary predictions for the CMB, arising from an earlier pre-inflationary era, some of the pre-inflationary nonequilibrium modes must enter the Hubble radius, and they must avoid complete relaxation by the time inflation begins. Because pre-inflationary modes with larger values of λ enter the Hubble radius later, they are presumably less likely to relax before inflation begins, in which case residual nonequilibrium will be possible only for λ larger than some infra-red cutoff λ_{c}. (For further discussion, see ref. [20].)

We hope that future work, based on a specific pre-inflationary model, will provide a prediction for λ_{c}, as well as some indication of the form of the nonequilibrium spectrum for $\lambda \gtrsim \lambda_{\mathrm{c}}$. Note that $\xi(k) < 1$ at wave number k implies that the nonequilibrium width of the corresponding inflaton mode is less than the equilibrium width. One might reasonably expect this, in view of the hypothesis that quantum noise arose from statistical relaxation processes in the very early universe: it seems natural to assume that early nonequilibrium would have a less-than-quantum dispersion, $\xi(k) < 1$, as opposed to a larger-than-quantum dispersion, $\xi(k) > 1$ (though the latter is of course possible in principle). Thus, a dip $\xi(k) < 1$ in the power spectrum below some critical wave number $k_{\mathrm{c}} = 2\pi/\lambda_{\mathrm{c}}$ might be naturally explained in terms of quantum nonequilibrium surviving from a very early pre-inflationary era.

It has in fact been found that an infra-red cutoff in the primordial power spectrum provides a slightly better fit to the 3-year WMAP data; however, the improvement is not sufficient to justify introducing the additional cutoff parameter in the model [34].

10.2 Super-Hubble Correlations without Inflation?

As noted in the introduction, one motivation for assuming quantum nonequilibrium at the big bang was that the resulting nonlocality at early times could eliminate the cosmological horizon problem (which persists, as we have mentioned, even in some inflationary models [19]). One might also ask if early quantum nonequilibrium could provide an alternative, non-inflationary means of laying down primordial curvature perturbations at super-Hubble lengthscales, in a standard Friedmann cosmology. Since we have shown that nonequilibrium can remain frozen at super-Hubble scales, one may ask if such nonequilibrium could generate appropriate super-Hubble correlations without the need for an inflationary era.

The Bunch-Davies vacuum for a scalar field ϕ, with variance given by (58) (at long wavelengths), has the remarkable property that the two-point correlation function

$$\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle = \frac{1}{V}\int d^3\mathbf{k}\; e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\left\langle |\phi_{\mathbf{k}}|^2\right\rangle$$

is independent of distance $|\mathbf{x}_1-\mathbf{x}_2|$, as is readily verified for $\left\langle |\phi_{\mathbf{k}}|^2\right\rangle = \left\langle |\phi_{\mathbf{k}}|^2\right\rangle_{\mathrm{QT}} \propto 1/k^3$. As a first step, one may ask how this inflationary quantum behaviour could be mimicked by a non-inflationary vacuum in quantum nonequilibrium.

Consider a vacuum state whose quantum variance is $\left\langle |\phi_{\mathbf{k}}|^2\right\rangle_{\mathrm{QT}} \propto k^{m_{\mathrm{QT}}}$ for some fixed index m_{QT}. Assuming that the quantum two-point function decreases with distance, where $\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_{\mathrm{QT}} \propto |\mathbf{x}_1-\mathbf{x}_2|^{-(m_{\mathrm{QT}}+3)}$, we have $m_{\mathrm{QT}} > -3$. Then consider the same vacuum state in quantum nonequilibrium, with $\left\langle |\phi_{\mathbf{k}}|^2\right\rangle = \left\langle |\phi_{\mathbf{k}}|^2\right\rangle_{\mathrm{QT}}\xi(k)$, assuming that $\xi(k)\propto k^{\mu}$ for some fixed index μ. To obtain a nonequilibrium two-point function that is independent of distance, we require $m_{\mathrm{QT}}+\mu = -3$, or $\mu < 0$, so that (in this simple example) the nonequilibrium function $\xi(k)$ must increase as $k\to 0$.

As things stand, we are unable to say if such behaviour for $\xi(k)$ is likely to emerge from any reasonable model. However, given the upper bound (28) on the ratio $\langle|\delta q_{\mathbf{k}r}(t_f)|\rangle_{\mathrm{eq}}/\Delta q_{\mathbf{k}r}(t_f)$, one could study how the 'degree of freezing' varies with k (for example for $k\to 0$), where a high or low degree of freezing could be defined respectively as a low or high value of the upper bound on the right-hand-side of (28). For a specific cosmological model, with some assumptions about initial conditions, this could provide constraints on the behaviour of the function $\xi(k)$. The results will obviously depend on how $\langle \hat{n}_{\mathbf{k}r}\rangle$ varies with k.

Finally, we note that a nonlocal model, based not on hidden variables or quantum nonequilibrium but on the holographic principle, has been shown to generate the required (approximately) scale-invariant perturbation spectrum at super-Hubble scales [35]. Whether or not early quantum nonequilibrium could reproduce such effects in a natural way remains to be seen.

10.3 Relic Nonequilibrium Particles

We saw in section 10.1 that relic nonequilibrium field modes could, in the case of the inflaton, change the power spectrum for primordial curvature perturbations, resulting in observable effects in the CMB. Thus, inflationary cosmology provides a simple and definite means whereby early nonequilibrium could yield observable consequences today. In this section, in contrast, we shall attempt to outline some much more complicated and uncertain scenarios, according to which relic nonequilibrium field modes (for some appropriate field) might in some circumstances manifest as relic nonequilibrium particles that could be detected today. Unfortunately, for these scenarios to be at all plausible, some questionable assumptions have to be made, and at the time of writing it is not clear if these scenarios can really work in practice.

There is of course no preferred definition of particle states in quantum field theory on expanding space, except in the short-wavelength limit (where one recovers the usual Minkowski definition) [36]. The very notion of 'particles' is in fact highly ambiguous for modes of frequency lower than the typical inverse timescale over which the spacetime metric changes. In a cosmological setting, this means that there is no generally useful definition of quantum particle states at wavelengths larger than the Hubble radius. Thus, if we consider relic nonequilibrium field modes from a radiation-dominated era — where the bound (57) implies that such modes must have super-Hubble wavelengths — we must be careful not to interpret such modes too naively in terms of (quantum) particle states. However, pending a more precise treatment, one might reasonably assume that if such modes enter the Hubble radius at later times, then they will manifest as (approximately-defined) particle states in the usual sense.

One should also bear in mind that, generally speaking, excitations of super-Hubble modes will not be produced by the local processes of particle scattering and decay (which are not expected to be effective over lengthscales larger than the instantaneous Hubble radius H^{-1}). However, such excitations will of course be produced by the global effects of spatial expansion.

In order to maximise the chance of obtaining relic nonequilibrium particles that could be detected in practice (in particular, with energies that are not so low as to be completely out of range), we ought to try to minimise the lower bound on the mode wavelength defined by (35) or (57). This can be done by choosing the final time t_f to be as small as possible, subject to the constraint that further relaxation may be neglected for times later than the chosen value of t_f. Thus, one might take t_f to be the time t_{dec} at which the relevant particle species decouples. For one might reasonably assume that relaxation may be neglected (at all wavelengths) for $t > t_{\mathrm{dec}}$ — if the quantum states, defined post-decoherence, are such that the associated de Broglie velocity field is sufficiently simple (as occurs, for example, for energy eigenstates). For a super-Hubble mode at t_{dec} that becomes sub-Hubble at later times, it is then conceivable that any nonequilibrium present at the time t_{dec} could persist until much later. (A proper discussion of this scenario would require an analysis of decoherence before and after decoupling.)

If we make the above assumptions, the key question is then whether residual nonequilibrium field modes can exist at the time $t_f = t_{\rm dec}$. From (57), this is possible if the modes have physical wavelength (inserting the Boltzmann constant $k_{\rm B}$)

$$\lambda_{\rm phys}(t_{\rm dec}) > 4\pi H_{\rm dec}^{-1} \ln(k_{\rm B}T_i/k_{\rm B}T_{\rm dec}) \ , \tag{61}$$

where $H_{\rm dec}^{-1}$ and $T_{\rm dec}$ are respectively the Hubble radius and temperature at time $t_{\rm dec}$. We have $\lambda_{\rm phys}(t_{\rm dec}) = a_{\rm dec}\lambda$, where $a_{\rm dec} = T_0/T_{\rm dec}$ (with $T_0 \simeq 2.7$ K the temperature today). Assuming that decoupling occurs before the end of the radiation-dominated phase, we also have $H_{\rm dec}^{-1} = 2t_{\rm dec}$, where $t_{\rm dec}$ may be expressed in terms of $T_{\rm dec}$ using the standard temperature-time relation

$$t \sim (1 \text{ s}) \left(\frac{1 \text{ MeV}}{k_{\rm B}T}\right)^2 \ . \tag{62}$$

The lower bound (61) then becomes (inserting the speed of light c)

$$\lambda \gtrsim 8\pi c(1 \text{ s}) \left(\frac{1 \text{ MeV}}{k_{\rm B}T_{\rm dec}}\right) \left(\frac{1 \text{ MeV}}{k_{\rm B}T_0}\right) \ln\left(\frac{k_{\rm B}T_i}{k_{\rm B}T_{\rm dec}}\right) , \tag{63}$$

or (with $c \simeq 3\times 10^{10}$ cm s^{-1}, $k_{\rm B} \simeq 8.6\times 10^{-5}$ eV K^{-1}, and $k_{\rm B}T_0 \simeq 2.3\times 10^{-4}$ eV)

$$\lambda \gtrsim (3.3 \times 10^{21} \text{ cm}) \left(\frac{1 \text{ MeV}}{k_{\rm B}T_{\rm dec}}\right) \ln\left(\frac{k_{\rm B}T_i}{k_{\rm B}T_{\rm dec}}\right) . \tag{64}$$

This provides a lower bound on the wavelength λ *today*, at which nonequilibrium could be found.

The freezing inequality (29), and the resulting lower bound (64), have been derived in this paper for massless scalar fields only. One certainly expects to find comparable results for more general massless boson fields, such as the electromagnetic field. For fermions, however, a separate analysis is required. There are different approaches to the pilot-wave theory of fermions, and the details of nonequilibrium freezing may depend on which model is adopted. One might try to derive a fermionic analogue of the freezing inequality using, for example, the Dirac sea pilot-wave model [37]. Pending such extensions of our analysis, here we assume that the lower bound (64) applies (at least approximately) to fermions as well, provided they are effectively massless at the temperature $T_{\rm dec}$ (that is, of mass $m << k_{\rm B}T_{\rm dec}/c^2$).

With this understanding, let us now apply the approximate result (64) to various particle species, both bosonic and fermionic. For definiteness, we first consider a standard Friedmann cosmology with no inflationary period, taking our initial conditions at the Planck era $k_{\rm B}T_i \sim k_{\rm B}T_{\rm P} \sim 10^{19}$ GeV. (An alternative possibility, of nonequilibrium relic particles arising from the decay of the inflaton, is considered below.)

Photons decouple from matter at $k_{\rm B}(T_{\rm dec})_\gamma \sim 0.3$ eV. From (64) we then have a lower bound

$$\lambda_\gamma \gtrsim 0.7 \times 10^{30} \text{ cm} \ , \tag{65}$$

which exceeds the Hubble radius today, $H_0^{-1} \simeq 10^{28}$ cm. If instead we consider neutrinos, which decouple at $k_{\rm B}(T_{\rm dec})_\nu \sim 1$ MeV, we have

$$\lambda_\nu \gtrsim 1.7 \times 10^{23} \text{ cm} \simeq 5.5 \times 10^4 \text{ pc} \tag{66}$$

(or $\sim 10^5$ light years). Residual nonequilibrium for relic neutrinos could plausibly exist today only at such tiny energies. Unfortunately, this is of course far outside any realistic range of detection. (Note, again, the implicit assumption being made, that if nonequilibrium super-Hubble modes at $t_{\rm dec}$ enter the Hubble radius at $t > t_{\rm dec}$, they will manifest as nonequilibrium particle states.)

The situation improves drastically, however, if one considers particles that decouple soon after the Planck era. Gravitons, for example, are expected to decouple at a temperature $(T_{\rm dec})_g \lesssim T_{\rm P}$. Writing

$$k_{\rm B}(T_{\rm dec})_g \equiv x_g(k_{\rm B}T_{\rm P}) \simeq x_g(10^{19} \text{ GeV}) ,$$

where $x_g \lesssim 1$, we obtain

$$\lambda_g \gtrsim (0.3 \text{ cm})(1/x_g)\ln(1/x_g) . \tag{67}$$

This might be compared with the range of wavelengths expected for a (thermal) relic graviton background, whose temperature today is estimated to be $(T_0)_g \sim 1$ K [38]. At this temperature, the spectral energy density of a Planck distribution peaks at the wavelength $\lambda_{\rm max}(1 \text{ K}) \simeq 0.3$ cm.

There may also exist other particles that decouple not too long after the Planck era, and that (unlike the graviton) are unstable, eventually producing decay products that could be more easily detected today. A natural candidate, arising out of current supersymmetric theories of high-energy physics, is the unstable gravitino $\tilde{G}$, which has been estimated to decouple at a temperature [39]

$$k_{\rm B}(T_{\rm dec})_{\tilde{G}} \equiv x_{\tilde{G}}(k_{\rm B}T_{\rm P}) \approx (1 \text{ TeV}) \left(\frac{g_*}{230}\right)^{1/2} \left(\frac{m_{\tilde{G}}}{10 \text{ keV}}\right)^2 \left(\frac{1 \text{ TeV}}{m_{gl}}\right)^2 ,$$

where g_* is the number of spin degrees of freedom (for the effectively massless particles) at the temperature $(T_{\rm dec})_{\tilde{G}}$, m_{gl} is the gluino mass, and $m_{\tilde{G}}$ is the gravitino mass. This provides us with an estimate for the lower bound in the case of gravitinos,

$$\lambda_{\tilde{G}} \gtrsim (0.3 \text{ cm})(1/x_{\tilde{G}})\ln(1/x_{\tilde{G}}) . \tag{68}$$

For the purposes of illustration, if we take $(g_*/230)^{1/2} \sim 1$ and $(1 \text{ TeV}/m_{gl})^2 \sim 1$, then

$$x_{\tilde{G}} \approx \left(\frac{m_{\tilde{G}}}{10^3 \text{ GeV}}\right)^2 .$$

If, for example, $m_{\tilde{G}} \approx 100$ GeV, then $x_{\tilde{G}} \approx 10^{-2}$ and (68) yields $\lambda_{\tilde{G}} \gtrsim 140$ cm. This corresponds to energies that are rather low, but perhaps accessible.

If the gravitino is not the lightest supersymmetric particle, then it will indeed be unstable. For large $m_{\tilde{G}}$, the total decay rate is estimated to be [40] $\Gamma_{\tilde{G}} =$

$(193/48)(m_{\tilde{G}}^3/M_{\rm P}^2)$, where $M_{\rm P} \simeq 1.2 \times 10^{19}$ GeV is the Planck mass. The time $(t_{\rm decay})_{\tilde{G}}$ at which the gravitino decays is of order the lifetime $1/\Gamma_{\tilde{G}}$. Using (62), the corresponding temperature is

$$k_{\rm B}(T_{\rm decay})_{\tilde{G}} \sim (m_{\tilde{G}}/1\ {\rm GeV})^{3/2}\ {\rm eV}\ .$$

For example, again for the case $m_{\tilde{G}} \approx 100$ GeV, the relic gravitinos decay when $k_{\rm B}(T_{\rm decay})_{\tilde{G}} \sim 1$ keV. This is prior to photon decoupling, so that any (potentially nonequilibrium) photons produced by the decaying gravitinos would interact strongly with matter and quickly relax to quantum equilibrium. To obtain gravitino decay after photon decoupling, we would need $k_{\rm B}(T_{\rm decay})_{\tilde{G}} \lesssim k_{\rm B}(T_{\rm dec})_{\gamma} \sim 0.3$ eV, or $m_{\tilde{G}} \lesssim 0.5$ GeV. For such small gravitino masses, however, decoupling occurs at (roughly)

$$(T_{\rm dec})_{\tilde{G}} = x_{\tilde{G}} T_{\rm P} \approx \left(m_{\tilde{G}}/10^3\ {\rm GeV}\right)^2 T_{\rm P} \lesssim 10^{-7} T_{\rm P}$$

and (68) (with $x_{\tilde{G}} \lesssim 10^{-7}$) yields the much larger lower bound $\lambda_{\tilde{G}} \gtrsim 10^7$ cm. Thus, it may prove more promising to consider other decay products (that decouple prior to gravitino decay but for larger gravitino masses). These could in turn decay into photons at later times, or they might be detected directly.

There are of course strong constraints on the presence of gravitinos in cosmological models, in particular from the abundance of light elements emerging from big-bang nucleosynthesis and from limits on dark matter abundance. These constraints have been extensively studied — see, for example, ref. [41] — and the subject is an active area of current research. Our hope is that an acceptable and compelling scenario will eventually be found, satisfying the standard cosmological constraints and at the same time allowing the possibility of relic nonequilibrium surviving in particles that could be detected today. To develop such a scenario in detail is a topic for future work.

So far in this section, we have assumed a standard (non-inflationary) Friedmann expansion, with initial nonequilibrium at around the Planck era. An alternative scenario is obtained if we consider relic nonequilibrium particles in the context of inflationary cosmology. If inflation did occur, the density of any relic particles (nonequilibrium or otherwise) from a pre-inflationary era will of course be so diluted as to be completely undetectable today. However, one may consider relic particles that were created at the end of inflation, by the decay of the inflaton field itself.

As discussed in section 10.1, during inflation the inflaton field does not relax to quantum equilibrium, and in fact the exponential expansion of space transfers any initial nonequilibrium from microscopic to macroscopic lengthscales. The inflaton field, then, is a prime candidate for a carrier of primordial quantum nonequilibrium. As well as manifesting as statistical anomalies in the CMB, such nonequilibrium in the inflaton field could manifest as nonequilibrium in its decay products, where in standard inflationary scenarios inflaton decay is in fact the source of the matter and radiation present in our universe today.

The process of 'preheating' is driven by the homogeneous and essentially classical part of the inflaton field (that is, by the $k = 0$ mode) [42]. Here, the

inflaton is treated as a classical external field, acting on other (quantum) fields which become excited by parametric resonance. Because of the classicality of the relevant part of the inflaton field, this process is unlikely to result in a transference of nonequilibrium from the inflaton to the created particles.

During 'reheating', however, perturbative decay of the inflaton can occur, and one may reasonably expect nonequilibrium in the inflaton field to be transferred to its decay products. This possibility opens up a large field of investigation. Here, again, we restrict ourselves to making some preliminary remarks.

The perturbative decay of the inflaton occurs through local field-theoretical interactions, so one expects the decay products to have physical wavelengths no greater than the instantaneous Hubble radius. Taking the lower bound (57) as a guide (even though it was derived for a radiation-dominated phase), we then expect that the decay products will come into existence already violating the freezing inequality. Subsequent relaxation might then be avoided (possibly) only if the particles are created at a temperature below their decoupling temperature. Once again, the gravitino suggests itself as a possible candidate. Gravitinos can in fact be copiously produced by inflaton decay [43] (and could even make up a significant component of dark matter [44]). If the gravitinos are unstable, again, one could try to detect (say) photons produced by their decay at later times.

The possible realisation of this scenario depends of course on uncertain features of high-energy particle physics and of inflationary models. As before, one may hope that a scenario will eventually be found, satisfying the constraints of particle physics and cosmology, and at the same time allowing the possibility of relic nonequilibrium surviving in particles that could be detected today.

We close this section with some general remarks.

First, we note that particle decay (for example for the gravitino) is likely to result in some relaxation and erasure of any quantum nonequilibrium that may have existed in the parent particles. However, one hopes that the erasure will not be complete and that some nonequilibrium will still be present in the decay products. It would be useful to study this, in pilot-wave models of specific decay processes.

Second, once suitable candidates for nonequilibrium relic particles have been identified, one must consider how best to test them for violations of the Born rule. For photons, a particularly simple test involves searching for anomalous polarisation probabilities, or deviations from Malus' law (where such deviations reflect the nonequilibrium breakdown of expectation additivity for non-commuting quantum observables in a two-state system) [12, 14].

Third, for a given species of relic particle in the universe today, even if there exist pure subensembles with significant residual nonequilibrium, in practice it might be difficult for us to locate those subensembles and perform experiments with them. In particular, if a given detector registers particles belonging to different subensembles, without distinguishing between them, it is possible that even if nonequilibrium is present in the individual subensembles it will not be visible in the data.

11 Conclusion

The hypothesis of quantum nonequilibrium at the big bang has been shown to have a number of observable consequences. Our main result is the freezing inequality (29). For cosmological field modes satisfying (29), initial nonequilibrium will be 'frozen' at later times. This result may be applied to specific cosmological models, yielding predictions whose verification could constitute evidence for quantum nonequilibrium in our universe. For a radiation-dominated expansion, (29) implies the general lower bound (35) on the wavelength of relic nonequilibrium field modes.

The detailed study of quantum nonequilibrium freezing, for realistic cosmological models, is left for future work. A useful first step might be to study the system of equations (43), and to delineate the general conditions under which the time evolution of a (mean) mode occupation number $\langle \hat{n}_{\mathbf{k}r} \rangle_t$ can satisfy the freezing inequality (29). Crucially, future work will need to study the statistical distribution of wave functionals for a realistic mixed state on expanding space, the goal being to identify subensembles satisfying (29). For these subensembles, quantum nonequilibrium is expected to be frozen over the relevant time period, resulting in definite predictions that might be tested today.

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